MAU22200 2021-2022 Practice Problem Set 1

The following is a theorem in Real Analysis:

> Suppose f and g are functions from $\mathbf{R}$ to $\mathbf{R}$, that $w \in \mathbf{R}$ and that $z \in \mathbf{R}$. Suppose further that $f(x) \leq g(x)$ for all $x \in \mathbf{R}$ and that $\lim_{x\to w} f(x)$ and $\lim_{x\to w} g(x)$ exist. Then $\lim_{x\to w} f(x) \leq \lim_{x\to w} g(x)$.

Here is a proof:

> Suppose $\lambda > 0$. Let $\epsilon = \lambda/2$. Then $\epsilon > 0$. Let
>
> $$z_1 = \lim_{x\to w} f(x)$$
>
> and
>
> $$z_2 = \lim_{x\to w} g(x).$$
>
> By the definition of limits of real valued functions of a real variable there is a $\delta_1 > 0$ such that if $0 < |x - w| < \delta_1$ then $|f(x) - z_1| < \epsilon$ and a $\delta_2 > 0$ such that if $0 < |x - w| < \delta_2$ then $|g(x) - z_2| < \epsilon$. Let
>
> $$\delta = \min(\delta_1, \delta_2).$$
>
> Set $x = w + \delta/2$. Then $0 < |x - w| < \delta$ so $0 < |x - w| < \delta_1$ and $0 < |x - w| < \delta_2$. It follows that $|f(x) - z_1| < \epsilon$ and $|g(x) - z_2| < \epsilon$. Therefore
>
> $$z_1 < f(x) + \epsilon \leq g(x) + \epsilon < z_2 + 2\epsilon = z_2 + \lambda.$$
>
> So $\lim_{x\to w} f(x) \leq \lambda + \lim_{x\to w} f(g)$ for all positive λ, which is possible only if $\lim_{x\to w} f(x) \leq \lim_{x\to w} f(g)$.

1. State and prove the corresponding theorem for functions from $\mathbf{R}^m$ to $\mathbf{R}$, where $m > 0$.

2. State and prove the corresponding theorem for functions from a normed vector space to $\mathbf{R}$.

3. State and prove the corresponding theorem for functions from a subset of a metric space to $\mathbf{R}$.