MAU22200 2021-2022 Practice Assignment 4, Due 14 April 2022

1. Define the *density* of a measurable subset E of $\mathbf{R}$ at a point $x \in \mathbf{R}$ to be

$$\lim_{h \searrow 0} \frac{1}{2h} m([x-h, x+h] \cap E),$$

if this limit exists. Is there a set E whose density at x is $1/2$ for almost all $x \in \mathbf{R}$?

2. Our theory of integration in $\mathbf{R}^3$ is based on the fact that the volume of a tetrahedron with vertices at the points (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

$$\frac{1}{3!} \left| \det \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{bmatrix} \right) \right|.$$

Show that this formula behaves as expected under changes of coordinates, i.e. that if (x_0', y_0', z_0'), (x_1', y_1', z_1'), (x_2', y_2', z_2') and (x_3', y_3', z_3') are related to (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) by a symmetry of the Euclidean space $\mathbf{R}^3$ then

$$\frac{1}{3!} \left| \det \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_0' & x_1' & x_2' & x_3' \\ y_0' & y_1' & y_2' & y_3' \\ z_0' & z_1' & z_2' & z_3' \end{bmatrix} \right) \right| = \frac{1}{3!} \left| \det \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{bmatrix} \right) \right|.$$

The symmetries of $\mathbf{R}^3$ are

$$\begin{aligned} x' &= q_{1,1}x + q_{1,2}y + q_{1,3}z + a \\ y' &= q_{2,1}x + q_{2,2}y + q_{2,3}z + b \\ z' &= q_{3,1}x + q_{3,2}y + q_{3,3}z + c \end{aligned}$$

where Q is an orthogonal matrix.
Hint: Matrices are easier to work with than coordinates. Also, there's nothing special about $\mathbf{R}^3$. I just chose $n = 3$ to make things concrete but you're better off not using any special properties of $\mathbf{R}^3$.

3. Consider the following "proof" that all bounded continuous functions on $\mathbf{R}$ are zero.

Suppose h is bounded and continuous. Let

$$f_n(x) = \frac{1}{\pi} \int_{y \in \mathbf{R}} \frac{nh(y)}{1 + n^2(x-y)^2} \, dm(y).$$

We evaluate

$$\lim_{n \to \infty} f_n(x)$$

in two different ways. First of all, for any $y \neq x$ we have

$$\lim_{n\to\infty} \frac{nh(y)}{1+n^2(x-y)^2} = 0.$$

$\{x\}$ is of measure zero so

$$\lim_{n\to\infty} \frac{nh(y)}{1+n^2(x-y)^2} = 0$$

for almost all y and hence

$$\int_{y\in\mathbf{R}} \lim_{n\to\infty} \frac{nh(y)}{1+n^2(x-y)^2}\, dm(y) = \int_{y\in\mathbf{R}} 0\, dm(y) = 0.$$

Exchanging the limit and integral and dividing by π, we get

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{y\in\mathbf{R}} \frac{nh(y)}{1+n^2(x-y)^2}\, dm(y) = 0.$$

In other words,

$$\lim_{n\to\infty} f_n(x) = 0.$$

On the other hand, we can make the change of variable

$$y = x + u/n$$

we see that

$$f_n(x) = \frac{1}{\pi} \int_{u\in\mathbf{R}} \frac{h(x+u/n)}{1+u^2}\, dm(u).$$

By the continuity of f we have

$$\lim_{n\to\infty} \frac{h(x+u/n)}{1+u^2} = \frac{h(x)}{1+u^2}.$$

Exchanging the limit and the integral and dividing by π we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\pi} \int_{u\in\mathbf{R}} \frac{h(x+u/n)}{1+u^2}\, dm(u) &= \frac{1}{\pi} \int_{u\in\mathbf{R}} \frac{h(x)}{1+u^2}\, dm(u) \\
&= h(x) \frac{1}{\pi} \int_{u\in\mathbf{R}} \frac{1}{1+u^2}\, dm(u) \\
&= h(x).
\end{aligned}$$

So

$$\lim_{n\to\infty} f_n(x) = h(x).$$

Combining this with our other method of evaluating the limit we find

$$h(x) = 0.$$

So all bounded continuous functions are zero.

Of course it's not difficult to give examples of bounded continuous functions which are not zero, so the argument above must have a fundamental flaw.

It has gaps, of course, which could be filled in. For example, one needs to evaluate

$$\int_{u\in\mathbf{R}} \frac{1}{1+u^2}\,dm(u).$$

This can be done as follows. By the Monotone Convergence Theorem we have

$$\begin{aligned}\lim_{k\to\infty}\int_{u\in\mathbf{R}} \frac{\chi_{[-k,k]}(u)}{1+u^2}\,dm(u) &= \int_{u\in\mathbf{R}} \lim_{k\to\infty}\frac{\chi_{[-k,k]}(u)}{1+u^2}\,dm(u)\\ &= \int_{u\in\mathbf{R}} \frac{1}{1+u^2}\,dm(u).\end{aligned}$$

Now

$$\int_{u\in\mathbf{R}} \frac{\chi_{[-k,k]}(u)}{1+u^2}\,dm(u) = \int_{u\in[-k,k]} \frac{1}{1+u^2}\,dm(u).$$

For continuous functions on a closed interval the Riemann and Lebesgue integrals both exist and agree, so

$$\int_{u\in[-k,k]} \frac{1}{1+u^2}\,dm(u) = \int_{-k}^{k} \frac{1}{1+u^2}\,du.$$

This integral can be evaluated using the Second Fundamental Theorem of Calculus, since $\frac{1}{1+u^2}$ is the derivative of $\arctan u$. This gives

$$\int_{-k}^{k} \frac{1}{1+u^2}\,du = \arctan(k) - \arctan(-k) = 2\arctan k.$$

Combining everything above shows that

$$\int_{u\in\mathbf{R}} \frac{1}{1+u^2}\,dm(u) = \lim_{k\to\infty} 2\arctan(k) = \pi.$$

So the step where I said that

$$h(x)\frac{1}{\pi}\int_{u\in\mathbf{R}} \frac{1}{1+u^2}\,dm(u) = h(x)$$

had a gap, but not an actual error, since the gap can be filled. There are various other gaps, in the sense of steps where details are missing, and all but one of those can gaps can be filled. One of them is an actual error though, in the sense that no filling is possible. Where is the error?