MAU22200 2021-2022 Practice Assignment 3, Due 1 March 2022
Solutions

1. $\ell^p(\mathbf{N})$ was defined for $p \in [1, +\infty)$ as the space of sequences $\alpha\colon \mathbf{N} \to \mathbf{R}$ such that
$$\sum_{j=0}^{n} |\alpha_j|^p$$
converges[1], equipped with the norm
$$\|\alpha\|_p = \left(\sum_{j=0}^{n} |\alpha_j|^p\right)^{1/p}.$$
It was shown in the notes that this is indeed a norm.

   (a) It's usual to define $\ell^\infty(\mathbf{N})$ as the space of bounded sequences with the norm
$$\|\alpha\|_\infty = \sup_{j\in\mathbf{N}} |\alpha_j|.$$
Although the connection with the ℓ^p spaces for $p < +\infty$ is not obvious from the definitions it is in fact true that $\lim_{p\to\infty} \|\alpha\|_p = \|\alpha\|_\infty$. You don't need to prove this however. Instead prove that
$$\|\alpha\|_\infty = \sup_{j\in\mathbf{N}} |\alpha_j|$$
is in fact a norm.
Solution: We need to check the three conditions which define a norm. The supremum of a set of non-negative numbers is non-negative so $\|\alpha\|_\infty \geq 0$ for all $\alpha \in \ell^\infty(\mathbf{N})$. If $\alpha \neq 0$ then $\alpha_k \neq 0$ for some $k \in \mathbf{N}$, so
$$\|\alpha\|_\infty = \sup_{j\in\mathbf{N}} |\alpha_j| > |\alpha_k| > 0.$$
If $\alpha = 0$ then $\|\alpha\|_\infty = 0$, so $\|\alpha\|_\infty > 0$ if and only if $\alpha \neq 0$. This establishes the first property of norms.
If $\alpha \in \ell^\infty(\mathbf{N})$ and $\lambda \in \mathbf{R}$ then
$$\|\lambda\alpha\|_\infty = \sup_{j\in\mathbf{N}} |\lambda\alpha_j| = \sup_{j\in\mathbf{N}} |\lambda||\alpha_j| = |\lambda| \sup_{j\in\mathbf{N}} |\alpha_j| = |\lambda|\|\alpha\|_\infty.$$
This is the second property of norms.
If $\alpha, \beta \in \ell^\infty(\mathbf{N})$ then
$$\begin{aligned}\|\alpha+\beta\|_\infty &= \sup_{j\in\mathbf{N}} |\alpha_j + \beta_j| \leq \sup_{j\in\mathbf{N}} (|\alpha_j| + |\beta_j|) \\ &\leq \sup_{j\in\mathbf{N}} |\alpha_j| + \sup_{j\in\mathbf{N}} |\beta_j| = \|\alpha\|_\infty + \|\beta\|_\infty.\end{aligned}$$
This establishes the third and last property of norms.

[1] By this I mean converges in $\mathbf{R}$. Equivalently, these are the sequences such that $\sum_{j=0}^{n} |\alpha_j|^p < +\infty$ in $[-\infty, +\infty]$.

(b) Show that for $p \in (0,1)$

$$\|\alpha\|_p = \left(\sum_{j=0}^{n} |\alpha_j|^p \right)^{1/p}$$

is not a norm on the space of sequences such that

$$\sum_{j=0}^{n} |\alpha_j|^p$$

converges.
Solution: Define α and β by

$$\alpha_j = \begin{cases} 1 & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha \neq 0, \end{cases} \qquad \beta_j = \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{if } \alpha \neq 1. \end{cases}$$

Then $\alpha \in \ell^\infty(\mathbf{N})$, $\beta \in \ell^\infty(\mathbf{N})$, $\alpha + \beta \in \ell^\infty(\mathbf{N})$, $\|\alpha\|_\infty = 1$ and $\|\beta\|_\infty = 1$, but

$$\|\alpha + \beta\|_\infty = 2^{1/p} > 2 = \|\alpha\|_\infty + \|\beta\|_\infty$$

so the third property of norms is violated.

2. Suppose that $\alpha \colon \mathbf{N} \to \mathbf{R}$ and $\beta \colon \mathbf{N} \to \mathbf{R}$ are sequences such that $\sum_{i \in \mathbf{N}} \alpha_i$ and $\sum_{j \in \mathbf{N}} \beta_j$ converge. Show that $\sum_{k \in \mathbf{N}} \gamma_k$ converges, where

$$\gamma_k = \sum_{i=0}^{k} \alpha_i \beta_{k-i}$$

and that

$$\sum_{k \in \mathbf{N}} \gamma_k = \left(\sum_{i \in \mathbf{N}} \alpha_i \right) \left(\sum_{j \in \mathbf{N}} \beta_j \right)$$

Note: These are sums in the more general sense considered in Chapter 6 of the notes, not series. The corresponding result for series isn't true without additional hypotheses.
Hint: As discussed in Lecture 34, it's often better to use theorems than definitions.
Solution: By Proposition 6.2.4

$$\sum_{i \in \mathbf{N}} |\alpha_i| < +\infty$$

and

$$\sum_{j \in \mathbf{N}} |\beta_j| < +\infty.$$

By Tonelli's Theorem, Theorem 6.4.3, then

$$\sum_{(i,j)\in\mathbf{N}^2} |\alpha_i\beta_j| < +\infty.$$

By Proposition 6.2.3 then

$$\sum_{(i,j)\in\mathbf{N}^2} \alpha_i\beta_j$$

converges. By Fubini's Theorem, Theorem 6.4.4, we have

$$\sum_{(i,j)\in\mathbf{N}^2} \alpha_i\beta_j = \sum_{i\in\mathbf{N}}\sum_{j\in\mathbf{N}} \alpha_i\beta_j = \sum_{i\in\mathbf{N}} \alpha_i \sum_{j\in\mathbf{N}} \beta_j$$

Let

$$S_k = \{(i,j) \in \mathbf{N}^2 \colon i + j = k.$$

Then $S_k \cap S_l = \varnothing$ if $k \neq l$ and $\bigcup_{k\in\mathbf{N}} S_k = \mathbf{N}^2$. Proposition 6.4.1, with S being $\mathbf{N}^2$ and $\mathcal{A}$ being the set of sets S_k for $k \in \mathbf{N}$ and $f((i,j)) = \alpha_i\beta_j$ gives

$$\sum_{(i,j)\in\mathbf{N}^2} \alpha_i\beta_j = \sum_{k\in\mathbf{N}} \sum_{(i,j)\in S_k} \alpha_i\beta_j.$$

But

$$\sum_{(i,j)\in S_k} \alpha_i\beta_j = \sum_{i=0}^{k} \alpha_i\beta_{k-i} = \gamma_k$$

so

$$\sum_{(i,j)\in\mathbf{N}^2} \alpha_i\beta_j = \sum_{k\in\mathbf{N}} \gamma_k.$$

Therefore

$$\sum_{k\in\mathbf{N}} \gamma_k = \sum_{i\in\mathbf{N}} \alpha_i \sum_{j\in\mathbf{N}} \beta_j.$$

3. Suppose F is a countable subset of $\mathbf{R}$, $\mathcal{B}$ is the Borel algebra on $\mathbf{R}$ and $\mathcal{J}$ is the Jordan algebra on $\mathbf{R}$.

   (a) Show that $F \in \mathcal{B}$.
   Solution: For every $x \in F$ the set $\mathbf{R} \setminus \{x\} = (-\infty, x) \cup (x, +\infty)$ is open and hence Borel, therefore its complement, $\{x\}$ is also Borel. But then

   $$F = \bigcup_{x\in F} \{x\}$$

   is a countable union of Borel sets and therefore also a Borel set.

   (b) Show that if $F \in \mathcal{J}$ then F is bounded.
   Hint: What are $\mu^-(F)$ and $\mu^+(F)$?
   Solution: Suppose $E \in \mathcal{I}$, where $\mathcal{I}$ is, as usual, the set of finite unions of intervals, and $E \subseteq F$. E is then a finite union of intervals. The

only countable intervals are the empty and the singletons $\{x\}$, which have length 0, so $\mu(E) = 0$. Taking the supremum over all such E we have

$$\mu^-(F) = \sup_{\substack{E \in \mathcal{I} \\ E \subseteq F}} \mu(E) = 0$$

since $\mathcal{J}$ was defined as the completion of $\mathcal{I}$. It then follows that

$$\inf_{\substack{G \in \mathcal{I} \\ F \subseteq G}} \mu(G) = \mu^+(F) = \mu^-(F) = 0.$$

There is therefore a $G \in \mathcal{I}$ such that $F \subseteq G$ and $\mu(G) < +\infty$. G is therefore a finite union of bounded intervals and so is bounded. But then F, which is a subset of G, must also be bounded.

(c) Give an example of a bounded F such that $F \notin \mathcal{J}$.
Solution:

$$F = [0,1] \cup \mathbf{Q}$$

works. The argument is almost identical to the one given in Proposition 7.4.5 to show that $\mathbf{Q} \notin \mathcal{J}$. Between any two rationals in $[0,1]$ there is an irrational, also in $[0,1]$, and between any two irrationals in $[0,1]$ there is a rational in $[0,1]$. It follows that neither F nor $[0,1] \setminus F$ can contain an interval of positive length and therefore that $\mu(E) = 0$ for any $E \in \mathcal{J}$ such that $E \subseteq F$ or $E \subseteq [0,1] \setminus F$. If $F \in \mathcal{J}$ then $[0,1] \setminus F \in \mathcal{J}$ as well and $\mu(F) = \mu^-(F) = 0$ and $\mu([0,1] \setminus F) = \mu^-([0,1] \setminus F) = 0$. But then

$$1 = \mu([0,1]) = \mu(F) + \mu([0,1] \setminus F) = 0 + 0 = 0,$$

which is clearly false, so the assumption that $F \in \mathcal{J}$ is incorrect.