MAU22200 2021-2022 Practice Assignment 3, Due 1 March 2022

1. $\ell^p(\mathbf{N})$ was defined for $p \in [1, +\infty)$ as the space of sequences $\alpha\colon \mathbf{N} \to \mathbf{R}$ such that

$$\sum_{j=0}^{n} |\alpha_j|^p$$

converges[1], equipped with the norm

$$\|\alpha\|_p = \left(\sum_{j=0}^{n} |\alpha_j|^p \right)^{1/p} .$$

It was shown in the notes that this is indeed a norm.

   (a) It's usual to define $\ell^\infty(\mathbf{N})$ as the space of bounded sequences with the norm

   $$\|\alpha\|_\infty = \sup_{j \in \mathbf{N}} |\alpha_j|.$$

   Although the connection with the ℓ^p spaces for $p < +\infty$ is not obvious from the definitions it is in fact true that $\lim_{p\to\infty} \|\alpha\|_p = \|\alpha\|_\infty$. You don't need to prove this however. Instead prove that

   $$\|\alpha\|_\infty = \sup_{j \in \mathbf{N}} |\alpha_j|$$

   is in fact a norm.

   (b) Show that for $p \in (0, 1)$

   $$\|\alpha\|_p = \left(\sum_{j=0}^{n} |\alpha_j|^p \right)^{1/p}$$

   is not a norm on the space of sequences such that

   $$\sum_{j=0}^{n} |\alpha_j|^p$$

   converges.

2. Suppose that $\alpha\colon \mathbf{N} \to \mathbf{R}$ and $\beta\colon \mathbf{N} \to \mathbf{R}$ are sequences such that $\sum_{i\in\mathbf{N}} \alpha_i$ and $\sum_{j\in\mathbf{N}} \beta_j$ converge. Show that $\sum_{k\in\mathbf{N}} \gamma_k$ converges, where

$$\gamma_k = \sum_{i=0}^{k} \alpha_i \beta_{k-i}$$

[1] By this I mean converges in $\mathbf{R}$. Equivalently, these are the sequences such that $\sum_{j=0}^{n} |\alpha_j|^p < +\infty$ in $[-\infty, +\infty]$.

and that

$$\sum_{k \in \mathbf{N}} \gamma_k = \left(\sum_{i \in \mathbf{N}} \alpha_i \right) \left(\sum_{j \in \mathbf{N}} \beta_j \right)$$

Note: These are sums in the more general sense considered in Chapter 6 of the notes, not series. The corresponding result for series isn't true without additional hypotheses.
Hint: As discussed in Lecture 34, it's often better to use theorems than definitions.

3. Suppose F is a countable subset of $\mathbf{R}$, $\mathcal{B}$ is the Borel algebra on $\mathbf{R}$ and $\mathcal{J}$ is the Jordan algebra on $\mathbf{R}$.

   (a) Show that $F \in \mathcal{B}$.

   (b) Show that if $F \in \mathcal{J}$ then F is bounded.
   Hint: What are $\mu^-(F)$ and $\mu^+(F)$?

   (c) Give an example of a bounded F such that $F \notin \mathcal{J}$.