MAU22200 2021-2022 Practice Assignment 2, Due 1 December 2021

Solutions

1. For each of the following subsets of $\mathbf{R}$ answer each of the following questions: Is it open? Is it closed? Is it Hausdorff? Is it connected? Is it compact? Is it bounded?
   Note: You're only asked to provide yes or no answers, not proofs.
   (a) $A = (-2, 1] \cup [-1, 2)$
   (b) $B = (-\infty, -1] \cup [1, +\infty)$
   (c) $C = \mathbf{Q} \cap [-1, 1]$
   (d) $D = (-2, 2) \cap [-1, 1]$.

   Solution:

	open?	closed?	Hausdorff?	connected?	compact?	bounded?
A	yes	no	yes	yes	no	yes
B	no	yes	yes	no	no	no
C	no	no	yes	no	no	no
D	no	yes	yes	yes	yes	yes

2. Suppose $A \subseteq B \subseteq C$ and $\mathcal{T}_C$ is a topology on C. Let $\mathcal{T}_B$ be the subspace topology on B as a subset of C and let $\mathcal{T}_A$ be the subspace topology on A as a subset of B. Is $\mathcal{T}_A$ also the subspace topology on A as a subset of C?
   Solution: Yes. Suppose U is open in the subspace topology on A as a subset of C. Then $U = A \cap W$ where $\in \mathcal{T}_C$. Let $V = B \cap W$. Then $V \in \mathcal{T}_B$ because it's the intersection of B with an element of $\mathcal{T}_C$. But then
   $$U = A \cap W = (A \cap B) \cap W = A \cap (B \cap W) = A \cap V$$
   so $U \in \mathcal{T}_A$ since it's the intersection of A with an element of $\mathcal{T}_B$.

   Suppose, conversely, that $U \in \mathcal{T}_A$. Then there is a $V \in \mathcal{T}_B$ such that $U = A \cap V$. There is also a $W \in \mathcal{T}_C$ such that $V = B \cap W$. Then
   $$U = A \cap V = A \cap (B \cap W) = (A \cap B) \cap W = A \cap W.$$
   So U is the intersection of A with an element of $\mathcal{T}_C$ and therefore belongs to the subspace topology on A as a subset of C.

3. Suppose $(X, \mathcal{T})$ is a topological space For $x \in X$ let $\mathcal{S}(x)$ be the set of sets A such that $x \in A$, $A \subseteq X$ and A is connected. Let
   $$B(x) = \bigcup_{A \in \mathcal{S}(x)} A.$$

(a) Show that $x \in B(x)$ and that $B(x)$ is connected.
Solution: $\{x\} \in \mathcal{S}(x)$ so $\{x\} \subseteq B(x)$ and hence $x \in B(x)$. Suppose U, V are open subsets of $B(x)$, in the subspace topology, such that $U \cap V = \varnothing$ and $U \cap V = B(x)$. $x \in B(x)$ so $x \in U$ or $x \in V$. Assume, without loss of generality, that $x \in U$. For each $A \in \mathcal{S}(x)$ we have

$$(A \cap U) \cap (A \cap V) = A \cap (U \cap V) = A \cap \varnothing = \varnothing$$

and

$$(A \cap U) \cup (A \cap V) = A \cap (U \cap V) = A \cap B(x) = A.$$

Also $A \cap U$ and $A \cap V$ are open sets in the subspace topology on A. $x \in A \cap U$ and A is connected so $A \cap V = \varnothing$. But

$$V = B(x) \cap V = \left(\bigcup_{A \in \mathcal{S}(x)} A \right) \cap V = \bigcup_{A \in \mathcal{S}(x)} (A \cap V) = \bigcup_{A \in \mathcal{S}(x)} \varnothing = \varnothing.$$

So if U, V are open subsets of $B(x)$ such that $U \cap V = \varnothing$ and $U \cap V = B(x)$ then at most one of U or V is non-empty. In other words, $B(x)$ is connected.

(b) Show that for all $x, y \in X$ either $B(x) = B(y)$ or $B(x) \cap B(y) = \varnothing$.
Hint: If $B(x) \cap B(y) = \varnothing$ then there's a $z \in B(x) \cap B(y)$. Try to show that $B(x) = B(z) = B(y)$.
Solution: Suppose $B(x) \cap B(y) \neq \varnothing$. Then there's a $z \in B(x) \cap B(y)$. $B(x)$ is connected and $z \in B(x)$ so $B(x) \in \mathcal{S}(z)$ and hence $B(x) \subseteq B(z)$. By the first part $x \in B(x)$ so $x \in B(z)$. $B(z)$ is connected so $B(z) \in \mathcal{S}(x)$ and hence $B(z) \subseteq B(x)$. We already have the reverse inequality so $B(z) = B(x)$. The same argument with x replaced by y everywhere shows that $B(z) = B(y)$. So $B(x) = B(y)$. We've just shown that if $B(x) \cap B(y) = \varnothing$ then $B(x) = B(y)$. In other words, either $B(x) = B(y)$ or $B(x) \cap B(y) = \varnothing$.

(c) Show that $B(x)$ is closed.
Hint: Try showing that the closure of $B(x)$ is connected and contains x.
Solution: Let $C(x) = \overline{B(x)}$. Suppose $C(x) = U \cap V$ where U and V are open in the subspace topology on $C(x)$ and $U \cap V = \varnothing$. Then $B(x) \cap U$ and $B(x) \cap V$ are open in the subspace topology on $B(x)$. Also

$$B(x) = B(x) \cap C(x) = B(x) \cap (U \cup V) = (B(x) \cap U) \cup (B(x) \cap V)$$

and

$$(B(x) \cap U) \cap (B(x) \cap V) = B(x) \cap (U \cap V) = B(x) \cap \varnothing = \varnothing.$$

As shown in the first part $B(x)$ is connected so either $B(x) \cap U$ or $B(x) \cap V$ must be empty. If $B(x) \cap U = \varnothing$ then $B(x) \subseteq X \setminus U$. Now

$X \setminus U$ is closed, so by the definition of the closure $C(x) \subseteq X \setminus U$. But $U \subseteq C(x)$ so $U = \varnothing$. Similarly if $B(x) \cap V = \varnothing$ then $V = \varnothing$. So if $C(x) = U \cap V$ where U and V are open and $U \cap V = \varnothing$ then at least one of them is empty. So $C(x)$ is connected. $x \in B(x)$ by the first part and $B(x) \subseteq C(x)$ to $x \in C(x)$. Therefore $C(x) \in \mathcal{S}(x)$. It follows that $C(x) \subseteq B(x)$. But we also have $B(x) \subseteq C(x)$ so $B(x) = C(x) = \overline{B}(x)$, so $B(x)$ is closed.

4. Suppose $(X, \mathcal{T})$ is a normal topological space and $K \subseteq U \subseteq X$ with K closed and U open. Show that there are closed L and open V such that $K \subseteq V \subseteq L \subseteq U$.
Solution: U is open so $X \setminus U$ is closed. $(X, \mathcal{T})$ is normal so there are open V and W such that $K \subseteq V$, $X \setminus U \subseteq W$ and $V \cap W = \varnothing$. Let $L = X \setminus W$. The statement $X \setminus U \subseteq W$ is equivalent to to the statement $X \setminus W \subseteq U$, i.e. $L \subseteq U$. Also, $V \cap W = \varnothing$ is equivalent to $V \subseteq X \setminus W$, i.e. $V \subseteq L$.