MAU22200 2021-2022 Practice Assignment 1, Due 20 October 2021
Solutions

1. Suppose (X, d) is a metric space. Define $e\colon X \times X \to \mathbf{R}$ by

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

(a) Show that e is a metric on X.
Hint: It is helpful first to establish the following two properties of the function $\varphi\colon [0, +\infty) \to [0, 1)$ defined by $\varphi(t) = t/(1 + t)$.

(i) $\varphi(t_1) < \varphi(t_2)$ if and only if $t_1 < t_2$.
(ii) $\varphi(t_1 + t_2) \le \varphi(t_1) + \varphi(t_2)$.

Solution: Following the hint, we prove the two statements above.

$$\varphi(t_2) - \varphi(t_1) = \frac{t_2 - t_1}{(1 + t_1)(1 + t_2)}.$$

The denominator is positive for all $t_1, t_2 \in [0, +\infty)$ so $\varphi(t_2) - \varphi(t_1) > 0$ if and only if $t_2 - t_1 > 0$. This establishes (i).

$$\varphi(t_1) + \varphi(t_2) - \varphi(t_1 + t_2) = \frac{t_1 t_2 (2 + t_1 + t_2}{(1 + t_1)(1 + t_2)(1 + t_1 + t_2)}.$$

The denominator is positive and the numerator is non-negative, so $\varphi(t_1) + \varphi(t_2) - \varphi(t_1 + t_2) \ge 0$. This establishes (ii).
We now check the various conditions from the definition.

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)} \ge 0$$

because $d(x, y) \ge 0$. If $x \ne y$ then the numerator is positive so $e(x, y) > 0$. This establishes (a).

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} = e(y, x)$$

because $d(x, y) = d(y, x)$. This establishes (b).

$$e(x, z) = \varphi(d(x, z)) \le \varphi(d(x, y) + d(y, z))$$

Because of (i) and the fact that $d(x, z) \le d(x, y) + d(y, z)$.

$$\varphi(d(x, y) + d(y, z)) \le \varphi(d(x, y)) + \varphi(d(y, z)) = e(x, y) + e(y, z)$$

because of (ii).

(b) Show that the topology of open sets with respect to e is the same as the topology of open sets with respect to d.

Solution: Suppose U is an open set for d. Then for each $x \in U$ there is an $r > 0$ such that $d(x, y) < r$ implies $y \in U$. Let $s = \varphi(r)$. If $e(x, y) < s$ then $\varphi(d(x, y)) < \varphi(r)$ so $d(x, y) < r$ and hence $y \in U$. $r > 0$ so $s > 0$. Thus there is, for every $x \in U$ an $s > 0$ such that if $e(x, y) < s$ then $y \in U$. In other words, U is an open set for e.

Suppose, conversely, that U is an open set for e. Then for each $x \in U$ there is a $t > 0$ such that if $e(x, y) < t$ then $y \in U$. Let $s = \min(1/2, t)$. If $e(x, y) < s$ then $e(x, y) < t$ and so $y \in U$. Let $r = s/(1 - s)$. Then $r \in [0, 1] \subseteq [0, +\infty)$. If $d(x, y) < r$ then

$$e(x, y) = \varphi(d(x, y)) < \varphi(r) = s$$

by (i) and hence $y \in U$. So for every $x \in U$ there is an $r > 0$ such that if $d(x, y) < r$ then $y \in U$. In other words, U is an open set for d.

2. (a) Show that for any function $f : X \to Y$ and any $A \in \wp(X)$ and $B \in \wp(Y)$ we have

$$f_*(f^*(f_*(A))) = f_*(A)$$

and

$$f^*(f_*(f^*(B))) = f^*(B).$$

Solution: Suppose $y \in f_*(A)$. Then there is an $x \in A$ such that $f(x) = y$. $f(x) \in f_*(A)$ so $x \in f^*(f_*(A))$. $f(x) = y$ so $y \in f_*(f^*(f_*(A)))$. So $y \in f_*(f^*(f_*(A)))$ for each $y \in f_*(A)$. Therefore

$$f_*(A) \subseteq f_*(f^*(f_*(A))).$$

Suppose, conversely, that $y \in f_*(f^*(f_*(A)))$. Then $y = f(x)$ for some $x \in f^*(f_*(A))$. $x \in f^*(f_*(A))$ means $f(x) \in f_*(A)$. So there is a $w \in A$ such that $f(x) = f(w)$. But then $y = f(w)$ and $w \in A$ so $y \in f_*(A)$. So $y \in f_*(A)$ whenever $y \in f_*(f^*(f_*(A)))$. Therefore

$$f_*(f^*(f_*(A))) \subseteq f_*(A).$$

Combining the two inclusions above, we get

$$f_*(f^*(f_*(A))) = f_*(A).$$

Suppose that $x \in f^*(B)$. Then $f(x) \in B$. It follows that $f(x) \in f_*(f^*(B))$, and hence $x \in f^*(f_*(f^*(B)))$. So $x \in f^*(f_*(f^*(B)))$ whenever $x \in f^*(B)$ and hence

$$f^*(B) \subseteq f^*(f_*(f^*(B))).$$

Suppose, conversely, that $x \in f^*(f_*(f^*(B)))$, i.e. that $f(x) \in f_*(f^*(B))$. In that case there is a $w \in f^*(B)$ such that $f(w) = f(x)$. $w \in f^*(B)$ means $f(w) \in B$, so $f(x) \in B$ and hence $x \in f^*(B)$. So $x \in f^*(B)$ whenever $x \in f^*(f_*(f^*(B)))$ and hence

$$f^*(f_*(f^*(B))) \subseteq f^*(B).$$

Combining the two inclusions above, we get

$$f^*(f_*(f^*(B))) = f^*(B).$$

(b) Is it true in general that

$$f^*(f_*(A))) = A$$

and

$$f_*(f^*(B))) = B?$$

In each case provide a proof or a counter-example to justify your answer.

Solution: Neither is true. Let $f\colon \{a,b\} \to \{c\}$ be defined by $f(a) = f(b) = c$ and let $A = \{a\}$. Then $f_*(A) = \{c\}$ and $f^*(f_*(A)) = \{a,b\}$. Let $f\colon \{a\} \to \{b,c\}$ be defined by $f(a) = b$ and let $B = \{b,c\}$. Then $f^*(B) = \{a\}$ and $f_*(f^*(B)) = \{b\}$.

3. Suppose that $(D, \preccurlyeq)$ and $(E, \preccurlyeq)$ are directed sets and $\tau\colon D \to E$ is a monotone function such that for all $q \in E$ there is a $p \in D$ such that $q \preccurlyeq \tau(p)$. Suppose $(Y, \mathcal{T})$ is a topological space, $z \in Y$ and $f\colon E \to Y$ is a net such that $\lim f = z$. Show that $\lim f \circ \tau = z$.
Solution: Suppose $Z \in \mathcal{O}(z)$. By assumption, $\lim f = z$, so by the definition of limits of nets there is an $a \in E$ such that if $b \in E$ and $a \preccurlyeq b$ then $f(b) \in Z$. Let $q = a$. By assumption there is a $p \in D$ such that $q \preccurlyeq \tau(p)$. Suppose that $r \in D$ and $p \preccurlyeq r$. Then $a = q \preccurlyeq \tau(p)$, $\tau(p) \preccurlyeq \tau(r)$ because τ is monotone. $(E, \preccurlyeq)$ is a directed set so $a \preccurlyeq \tau(p)$ and $\tau(p) \preccurlyeq \tau(r)$ imply $a \preccurlyeq \tau(r)$. Therefore $f(\tau(r)) \in Z$. So if $r \in D$ and $p \preccurlyeq r$ then $(f \circ \tau)(r) \in Z$. There is such a $p \in D$ for each $Z \in \mathcal{O}(z)$, so $\lim f \circ \tau = z$.