MAU22200 2021-2022 Practice Assignment 1, Due 20 October 2021

1. Suppose (X, d) is a metric space. Define $e\colon X \times X \to \mathbf{R}$ by

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

   (a) Show that e is a metric on X.
   Hint: It is helpful first to establish the following two properties of the function $\varphi\colon [0, +\infty) \to [0, 1)$ defined by $\varphi(t) = t/(1 + t)$.
      (i) $\varphi(t_1) < \varphi(t_2)$ if and only if $t_1 < t_2$.
      (ii) $\varphi(t_1 + t_2) \leq \varphi(t_1) + \varphi(t_2)$.

   (b) Show that the topology of open sets with respect to e is the same as the topology of open sets with respect to d.

2. (a) Show that for any function $f\colon X \to Y$ and any $A \in \wp(X)$ and $B \in \wp(Y)$ we have

$$f_*(f^*(f_*(A))) = f_*(A)$$

   and

$$f^*(f_*(f^*(B))) = f^*(B).$$

   (b) Is it true in general that

$$f^*(f_*(A))) = A$$

   and

$$f_*(f^*(B))) = B?$$

   In each case provide a proof or a counter-example to justify your answer.

3. Suppose that $(D, \preccurlyeq)$ and $(E, \preccurlyeq)$ are directed sets and $\tau\colon D \to E$ is a monotone function such that for all $q \in E$ there is a $p \in D$ such that $q \preccurlyeq \tau(p)$. Suppose $(Y, \mathcal{T})$ is a topological space, $z \in Y$ and $f\colon E \to Y$ is a net such that $\lim f = z$. Show that $\lim f \circ \tau = z$.