MAU22200 Lecture 65

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Exam structure

You have three hours. There are six questions, three on Semester 1 and three on Semester 2. You are asked to do two from each set of three. You still just need to pass overall though, not each semester separately.

That was the pattern last year as well. The year before there was a final assignment rather than a final exam. Before that there were separate exams for MA2223 and MAU2224.

The questions are broken up into 2-6 parts, with point values indicated. You're allowed to use results from earlier parts in doing later parts. This is still true if you got the earlier parts wrong or didn't even attempt them. There is very little direct testing of memory, e.g. "define a normal space" or "state Urysohn's Lemma" or "prove Fatou's Lemma for sums".

I prefer questions which test knowledge of theorems indirectly, and involve some "unseen" element.

A sample question

Here's a question from last year which is a good example.

1. (5 points) Suppose that $f_n \colon \mathbf{R}^d \to \mathbf{C}$ are absolutely integrable functions. Prove that $\left|\sum_{n=1}^{N} f_n(x)\right| \leq \sum_{n=1}^{\infty} |f_n(x)|$ for all N.
2. (10 points) With f_n as above, prove that if $\sum_{n=1}^{\infty} \int_{\mathrm{R}^d} |f_n(x)|\, dx < \infty$ then

$$\int_{\mathrm{R}^d} \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_{\mathrm{R}^d} f_n(x)\, dx.$$

3. (10 points) Show, by means of an example, that the equation above can fail if the hypothesis $\sum_{n=1}^{\infty} \int_{\mathrm{R}^d} |f_n(x)|\, dx < \infty$ is removed.

There are some differences of notation and terminology from last year to this year. Also, last year I did complex valued integrals but this year they're all real valued.

A sample question

Here's how that question would appear on this year's exam. It won't, of course, but if it did it would look like this:

1. (5 points) Suppose that $f_n \colon \mathbf{R}^d \to \mathbf{R}$ are integrable functions. Prove that $\left|\sum_{n=0}^{N} f_n(\mathbf{x})\right| \leq \sum_{n=0}^{\infty} |f_n(\mathbf{x})|$ for all N.
2. (10 points) With f_n as above, prove that if $\sum_{n=0}^{\infty} \int_{\mathrm{x}\in\mathrm{R}^d} |f_n(\mathbf{x})|\, dm(\mathbf{x}) < \infty$ then

$$\int_{\mathrm{x}\in\mathrm{R}^d} \sum_{n=0}^{\infty} f_n(\mathbf{x})\, dm(\mathbf{x}) = \sum_{n=0}^{\infty} \int_{\mathrm{x}\in\mathrm{R}^d} f_n(\mathbf{x})\, dm(\mathbf{x}).$$

3. (10 points) Show, by means of an example, that the equation above can fail if the hypothesis $\sum_{n=0}^{\infty} \int_{\mathrm{x}\in\mathrm{R}^d} |f_n(\mathbf{x})|\, dm(\mathbf{x}) < \infty$ is removed.

Also, the points would be 4, 8 and 8, since this year they add up to 20 per question.

Part 1

$$\left|\sum_{n=0}^{N} f_n(\mathbf{x})\right| \leq \sum_{n=0}^{\infty} |f_n(\mathbf{x})| \textit{ for all } N.$$

This part is just testing definitions and elementary properties.
$\left|\sum_{n=0}^{N} f_n(\mathbf{x})\right| \leq \sum_{n=0}^{N} |f_n(\mathbf{x})|$ is the triangle inequality in $\mathbf{R}$.
$\sum_{n=0}^{N} |f_n(\mathbf{x})| \leq \sum_{n=0}^{\infty} |f_n(\mathbf{x})|$ because this is a sum of non-negative terms. Combining them gives $\left|\sum_{n=0}^{N} f_n(\mathbf{x})\right| \leq \sum_{n=0}^{\infty} |f_n(\mathbf{x})|$
In addition to testing basic properties this part is meant as a hint about how to do the next part.

Part 2

With f_n as above, prove that if $\sum_{n=0}^{\infty} \int_{\mathrm{x} \in \mathrm{R}^d} |f_n(\mathbf{x})| \, dm(\mathbf{x}) < \infty$ then

$$\int_{\mathrm{x} \in \mathrm{R}^d} \sum_{n=0}^{\infty} f_n(\mathbf{x}) \, dm(\mathbf{x}) = \sum_{n=0}^{\infty} \int_{\mathrm{x} \in \mathrm{R}^d} f_n(\mathbf{x}) \, dm(\mathbf{x}).$$

This is meant to test knowledge of one of the main theorems. Which theorem? We want to exchange an infinite sum and an integral, and infinite sums are limits of finite sums. Exchanging finite sums and integrals is unproblematic so the issue is exchanging the limit and the integral. We have three theorems about limits and integrals: the Monotone Convergence Theorem, Fatou's Lemma and the Dominated Convergence Theorem. Only the first and last actually allow you to exchange a limit and an integral, so it's one of those.

Part 2, continued

We're hoping to use the fact that $\sum_{n=0}^{\infty} f_n(\mathbf{x}) = \lim_{N\to\infty} g_N(\mathbf{x})$ where $g_N(\mathbf{x}) = \sum_{n=0}^{N} f_n(\mathbf{x})$. The easy part is exchanging the finite sum and the integral (linearity):

$$\int_{\mathrm{x}\in\mathrm{R}^d} g_N(\mathbf{x})\,dm(\mathbf{x}) = \sum_{n=0}^{N} \int_{\mathrm{x}\in\mathrm{R}^d} f_n(\mathbf{x})\,dm(\mathbf{x}).$$

What's left to prove is

$$\int_{\mathrm{x}\in\mathrm{R}^d} \lim_{N\to\infty} g_N(\mathbf{x})\,dm(\mathbf{x}) = \lim_{N\to\infty} \int_{\mathrm{x}\in\mathrm{R}^d} g_N(\mathbf{x})\,dm(\mathbf{x}).$$

This looks like the conclusion of the Monotone Convergence Theorem or the Dominated Convergence Theorem. g_N isn't monotone unless the f's are all positive so we probably want to use the Dominated Convergence Theorem. For that we need an integrable f such that $|g_N(\mathbf{x})| \leq h(\mathbf{x})$ for all N and $\mathbf{x}$,

Part 2, conclusion

We need an integrable h such that $|g_N(\mathbf{x})| \leq h(\mathbf{x})$ for all N and $\mathbf{x}$, i.e.

$$\left| \sum_{n=0}^{N} f_n(\mathbf{x}) \right| \leq h(\mathbf{x}).$$

Where do we get such an h? From Part 1! $h(\mathbf{x}) = \sum_{n=0}^{\infty} |f_n(\mathbf{x})|$.
Why is it integrable? We need $\int_{\mathbf{x}\in\mathbb{R}^d} \sum_{n=0}^{\infty} |f_n(\mathbf{x})| \, dm(\mathbf{x}) < +\infty$.
We have $\sum_{n=0}^{\infty} \int_{\mathbf{x}\in\mathbb{R}^d} |f_n(\mathbf{x})| \, dm(\mathbf{x}) < +\infty$. If we can show that the sum and integral can be exchanged then we're done.
Isn't this the problem we started with? No, because this time the summands are non-negative so we *can* apply the Monotone Convergence Theorem!

Part 3

Show, by means of an example, that the equation above can fail if the hypothesis $\sum_{n=0}^{\infty} \int_{x \in \mathbb{R}^d} |f_n(\mathbf{x})| \, dm(\mathbf{x}) < \infty$ *is removed.*

I'm not sure I'd ask this part this year, although you've seen the counterexample, more or less, in Lecture 35.

$$\alpha_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\sum_{j=0}^{\infty} \alpha_{i,j} = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases} \qquad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{i,j} = 1.$$

$$\sum_{i=0}^{\infty} \alpha_{i,j} = \begin{cases} -1 & \text{if } j = -1, \\ 0 & \text{if } j \neq -1. \end{cases} \qquad \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \alpha_{i,j} = 0.$$

So $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{i,j} \neq \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \alpha_{i,j}$.

Part 3, continued

We need to convert a sum to an integral. Usually we try to convert integrals to sums, but here it's the other way around. The functions to take are

$$f_n(x) = \begin{cases} 1 & \text{if } n < x < n+1, \\ -1 & \text{if } n+1 < x < n+2, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_{x \in \mathrm{R}} \sum_{n=0}^{\infty} f_n(x)\, dm(x) = \int_{x \in \mathrm{R}} \chi_{(0,1)}(x)\, dm(x) = 1$$

while

$$\sum_{n=0}^{\infty} \int_{x \in \mathrm{R}} f_n(x)\, dm(x) = \sum_{n=0}^{\infty} 0 = 0.$$

Revision

There are a variety of things you can use for revision, each with some advantages and some disadvantages:

- Past exams: These are similar in that they are exams (except for 2019-2020), and the core of the content is fairly stable from year to year, but last year's was a 6 hour online exam, and everything before that was split into MA2223 and MA2224. I taught Semester 2 last year and Semester 1 the year before. I also taught MA2223 at one point and half of MAU2224 once, but mostly the past papers are by other people, who may ask different types of questions. Also, as you've just seen, there are some minor changes of terminology and notation from year to year. We don't generally give out solutions to past papers, although you can certainly ask about individual questions.
- Notes: These are fairly comprehensive, but very long! Also, the proofs are often longer or trickier than anything I'd ask for on an exam.

Revision, continued

- Assignments and practice problems: These are in this year's terminology and notation and are the sorts of questions I ask. Also, you have worked solutions! The problems are generally longer than I would ask on an exam though. I wasn't able to cover everything on the assignments and practice problems and I definitely can't cover everything on the exam. The overlap is largely random.
- Sample paper: I mostly try to avoid giving sample papers, but preparing these slides makes me think it might be helpful in this case. The reason I usually avoid them is that students learn how to do the questions on the sample paper. Don't do that! The way to use a sample paper is to see the *style* of questions likely to appear, not the topics and to get a sense for whether your *overall* understanding is detailed enough.