MAU22200 Lecture 64

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Plan for this week

- Monday 2pm (this lecture): General comments and module overview
- Monday 3pm (tutorial): as usual
- Tuesday 11am: The exam and how to revise for it.
- Tuesday 1pm: Question and answer session via Microsoft Teams. Ask questions at the session or email them in advance.

General comments

- You can't learn all of Mathematics!
- Your goal is to try to limit what you need to remember.
- There are some theorems (or definitions, examples, propositions, lemmas, etc.) that you do need to remember, but you don't need to know most of them.
- My goal is that you should know these theorems (or definitions, ...) and be able to use them in later modules. Ideally you should be able to recognise them in slightly unfamiliar contexts.
- Memorising proofs is largely a waste of time, at least for this module. That's not to say you should ignore them though. I'll say something about the uses of proofs later.

How to limit what you need to memorise (1/2)

- Often a later proposition makes an earlier one obsolete. For example, I proved successively stronger versions of Fubini's Theorem, but all are special cases of the last one, so you only need to remember that one.
- Often a proposition is only used to prove a theorem, and isn't of independent interest. How do you know which ones are important?
  - Things people need often tend to acquire names, e.g. Fubini's Theorem, the Dominated Convergence Theorem, Urysohn's Lemma, Markov's Inequality, The Cantor Set, etc. A name is a good indicator that something is important. It can also be used for communicating people outside this module. Even if you happen to remember what Proposition 9.5.2 of the notes said, no one else will.
  - Theorems are usually more important than propositions and propositions more important than lemmas, but this isn't very reliable, e.g. Fatou's Lemma is more important than the Tietze Extension Theorem.

How to limit what you need to memorise (2/2)

- Don't learn special cases separately. For example, all the convergence theorems for functions can be applied to characteristic functions to get statements for measures of sequences of sets. If you know this then you don't have to learn those consequences separately. There are exceptions though. $m_n(E \times F) = m_{n'}(E)m_{n''}(F)$ follows from Fubini's Theorem, but is worth knowing separately.

Why proofs?

I've given proofs for almost everything, but now I'm telling you you don't need to memorise them. Why did I give them?

- It keeps me honest. and it's what mathematicians do.
- A fact that gets used all the time in proofs in the notes is likely to be useful in your own proofs. How many times did I use, explicitly or implicitly, the linearity and monotonicity of integrals?
- They're often a source of useful tricks, e.g. the "nice plus small" idea from the proofs of the Lebesgue Differentiation Theorem and Fubini's Theorem is good to know.
- Sometimes you want a slight variant of a known theorem to be true. If you know the proof you can often make a good guess at whether it is true. For example, we didn't use much of the structure of $\mathbf{R}^n$ in proving Fubini, so you might guess, correctly, that there are more general versions.

Overview of first semester (1/3)

- Chapter 1 (Limits): This was mostly motivational in nature. Either it worked or it it didn't but most of the important bits reappear later. The main thing to retain is how a normed vector space is a metric space and how a metric space is a topological space. Also, the properties of images and preimages are used everywhere.
- Chapter 2 (Sets and Cardinality): This isn't a module on Set Theory, but the notions of finite, countable and uncountable are fundamental to Semester 2. The Cantor set often appears as an example or counter-example in Topology and in Measure and Integration.

Overview of first semester (2/3)

- Chapter 3 (Topological spaces): This is where things really begin. Topologies, interiors, closures, the Hausdorff property, continuity, subspace topologies, product topologies and compactness are used everywhere. Density, boundaries, weaker vs stronger topologies, quotient topologies, connectedness and normal spaces are not far behind.
- Chapter 4 (Metric spaces): Most of the topological spaces we actually meet are metric spaces, but often we only care about their topological properties. Some notions only make sense with a metric though: boundedness, Lipschitz and uniform continuity, Cauchy sequences/nets/filters, completion, etc. For some metric spaces we can characterise the compact sets explicitly, e.g. Heine-Borel for $\mathbf{R}^n$ or Arzelà-Ascoli for spaces of continuous functions.

Overview of first semester (3/3)

- Chapter 5 (Normed vector spaces): These are very important in more advanced analysis, e.g. Functional Analysis or Partial Differential Equations. We only cover their most basic properties in this module. ℓ^p is a useful source of examples. Most of Semester 2 could be done in the context of Banach spaces, i.e. complete normed vector spaces. We could have defined sums and integrals of functions with values in a Banach space, for example, but didn't.

Overview of second semester (1/4)

- Chapter 6 (Infinite sums): This chapter is partly a warm up for integration. We do need many of these results for proving the corresponding ones for integrals though. For example, the proof of the Monotone Convergence Theorem for integrals uses the Monotone Convergence Theorem for sums. Eventually, though, almost everything in this chapter becomes a special case (counting measure) of general results for integrals.
- Chapter 7 (Content and measure): This is basic to the theory of integration. The main point is really the definition and elementary properties of measures. For this of course you need σ-algebras. Most of the results here are trivial consequences of the definitions. The main exception is completion.

Overview of second semester (2/4)

- Chapter 8 (Integration): I've chosen to tie Riemann and Lebesgue integration closer together than is usual. This probably makes Riemann integration harder to understand and Lebesgue integration easier. The highlight of this chapter is the three convergence theorems in the last section.
- Chapter 9 (Constructing measures): This is mostly just a chapter on the Riesz Representation Theorem. As far as this module is concerned, the RRT is our only means of constructing non-trivial measures. There are other ways, but they're equally ugly. I like this one, because it concentrates on the integrals, which are what we ultimately want, rather than the contents and measures.

Overview of second semester (3/4)

- Chapter 10 (The Fundamental Theorem of Calculus): For Riemann integration the FTC is, well, fundamental. It's the main way of computing integrals of non-trivial functions. For Lebesgue integration it's a bit less fundamental, but still useful. Figuring out the correct hypotheses is much more complicated, as is proving the theorem. Ultimately we get both parts of the theorem from the corresponding results for Riemann integrals, plus various approximation arguments.
- Chapter 11 (Affine spaces and convex sets): If you're willing to believe that $\mathbf{R}^n$ has a well defined notion of area/volume/content for things which look like polygons/polyhedra then you can largely skip this chapter, and the first two sections of the next chapter.

Overview of second semester (4/4)

- Chapter 12 (Higher Dimensions): The main point of this chapter is show that Lebesgue integration in $\mathbf{R}^n$ is well defined for $n > 1$, and satisfies Fubini's Theorem. Fubini's Theorem gives a sufficient condition, integrability in the product space, for exchanging integrals. There are counter-examples if the condition is not satisfied. Tonelli's Theorem is more or less a by-product, but a useful one. They're often used in combination, with the conclusion of Tonelli used to show the hypothesis of Fubini is satisfied.