MAU22200 Lecture 63

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Recap (1/3)

We defined

$$I(g) = \int_{\mathbf{x} \in \mathbb{R}^n} g(\mathbf{x})\, dm_n(\mathbf{x})$$

$$I_1(g) = \int_{\mathbf{x}' \in \mathbb{R}^{n'}} \int_{\mathbf{x}'' \in \mathbb{R}^{n''}} g(\mathbf{x}', \mathbf{x}'')\, dm_{n''}(\mathbf{x}'')\, dm_{n'}(\mathbf{x}'),$$

$$I_2(g) = \int_{\mathbf{x}'' \in \mathbb{R}^{n''}} \int_{\mathbf{x}' \in \mathbb{R}^{n'}} g(\mathbf{x}', \mathbf{x}'')\, dm_{n'}(\mathbf{x}')\, dm_{n''}(\mathbf{x}'')$$

These had the following properties:

- I_j is linear.
- I_j is monotone, in the sense that if $f \leq g$ then $I_j(f) \leq I_j(g)$.
- There is a version of the Monotone Convergence Theorem for I_j.
- There is a version of the Dominated Convergence Theorem for I_j.

Recap (2/3)

We also defined $\mu_j(E) = I_j(\chi_E)$. This had the following properties:

- μ_j is monotone, i.e. $E \subseteq F$ implies $\mu_j(E) \leq \mu_j(F)$.
- μ_j is well behaved with respect to increasing sequences of sets, i.e. if $E_0 \subseteq E_1 \subseteq \cdots$ then

$$\mu_j\left(\bigcup_{k=0}^{\infty} E_k\right) = \lim_{k\to\infty} \mu_j(E_k)$$

- μ_j is well behaved with respect to decreasing sequences of sets, i.e. if $E_0 \supseteq E_1 \supseteq \cdots$ and $\mu_j(E_0) < +\infty$ then

$$\mu_j\left(\bigcap_{k=0}^{\infty} E_k\right) = \lim_{k\to\infty} \mu_j(E_k)$$

Recap (3/3)

I claimed the following were true:

- If K is compact then $\mu_j(K) \leq m_n(K)$.
- If U is open then $\mu_j(U) \geq m_n(U)$.
- If K is compact then $\mu_j(K) = m_n(K)$.
- If U is open then $\mu_j(U) = m_n(U)$.
- If E is Borel then $\mu_j(E) = m_n(E)$.
- If E is Lebesgue measurable then $\mu_j(E) = m_n(E)$.

I proved the first and claimed the proof of the second was similar. I still need to sketch the proof of the other four and explain how to use the last one to get Tonelli's Theorem and Fubini's Theorem.

Showing $\mu_j = m_n$ (1/5)

If K is compact then $\mu_j(K) = m_n(K)$.

This proof starts like the proof from last time: $m_n(K) < +\infty$. We can find a λ such that $m_n(K) < \lambda < +\infty$. Then we can find an open superset U of K such that $m_n(U) < \lambda$. Let

$$V_k = U \cap \bigcup_{\mathbf{x} \in K} B(\mathbf{x}, 1/2^k).$$

- V_k is open for each k, so $\mu_j(V_k) \geq m_n(V_k)$
- $V_0 \supseteq V_1 \supseteq \cdots$.
- $V_0 \subseteq U$ so $m_n(V_0) < +\infty$.
- $K = \bigcap_{k=0}^{\infty} V_k$. It's easy to see that $K \subseteq \bigcap_{k=0}^{\infty} V_k$. $\bigcap_{k=0}^{\infty} V_k \subseteq K$ is not much harder.

The decreasing sequence property of μ_j gives

$$\lim_{k \to \infty} \mu_j(V_k) \leq \mu_j\left(\bigcap_{k=0}^{\infty} V_k\right) = \mu_j(K).$$

Showing $\mu_j = m_n$ (2/5)

$$\lim_{k\to\infty} \mu_j(V_k) \leq \mu_j(K).$$

V_k is open for each k, so $\mu_j(V_k) \geq m_n(V_k)$. Therefore

$$\lim_{k\to\infty} m_n(V_k) \leq \mu_j(K).$$

$K \subseteq V_k$ so $m_n(K) \leq m_n(V_k)$. Therefore $m_n(K) \leq \mu_j(K)$. K is compact, so we got the reverse inequality last time. Therefore $\mu_j(K) = m_n(K)$, as claimed.

Showing $\mu_j = m_n$ (3/5)

If U is open then $\mu_j(U) = m_n(U)$.

The proof of this is similar, but with some changes. We apply the property of increasing sequences rather than decreasing sequences. We need an increasing sequence of compact sets whose union is U. One way to get one is to consider the set $\mathcal{A}$ of balls in U with rational centres and rational radii. There are countably many of them and the balls $\bar{B}(\mathbf{x}, r/2)$ for $B(\mathbf{x}, r) \in \mathcal{A}$ are compact and their union is U. They aren't an increasing sequence, but this is easily fixed. Set

$$K_l = \bigcup_{i=0}^{l} \bar{B}(\mathbf{x}_i, r_i/2).$$

Showing $\mu_j = m_n$ (4/5)

If E is Borel then $\mu_j(E) = m_n(E)$.

Roughly, we choose K compact and U open with $K \subseteq E \subseteq U$ and $m_n(U) < m_n(K) + \epsilon$. Then

$$\begin{aligned} m_n(E) - \epsilon \le m_n(U) - \epsilon &< m_n(K) = \mu_j(K) \\ &\le \mu_j(E) \le \mu_j(U) = m_n(U) \\ &< m_n(K) + \epsilon \le m_n(E) + \epsilon \end{aligned}$$

So $m_n(K) - \epsilon < \mu_j(E) < m_n(E) + \epsilon$ for all $\epsilon > 0$. Therefore $\mu_j(E) = m_n(E)$.
The proof above works if $m_n(E) < +\infty$. It requires some modifications to cover the case $m_n(E) = +\infty$.

Showing $\mu_j = m_n$ (5/5)

If E is Lebesgue measurable then $\mu_j(E) = m_n(E)$.

If E is Lebesgue measurable then there are Borel sets D and H such that $E \triangle H \subseteq D$ and $m_n(D) = 0$. This follows from the fact that Lebesgue measure is the completion of Borel measure. If these were contents rather than measures then we'd only have $m_n(D) < \epsilon$.

$$\begin{aligned} m_n(E) &\le m_n(H \cup D) = m_n(H \setminus D) + m_n(D) = m_n(H \setminus D) \\ &= \mu_j(H \setminus D) \le \mu_j(E) \le \mu_j(D \cup H) = m_n(D \cup H) \\ &\le m_n(H \setminus D) + m_n(D) = m_n(H \setminus D) \le m_n(E). \end{aligned}$$

The two ends are equal, so everything in between must be. Therefore $\mu_j(E) = m_n(E)$.

Tonelli and Fubini (1/4)

We then prove the following in sequence:

- If g is a simple function then $I_1(g) = I(g) = I_2(g)$.
- If g is a non-negative semisimple function then $I_1(g) = I(g) = I_2(g)$.
- If g is a non-negative measurable function then $I_1(g) = I(g) = I_2(g)$. This is known as *Tonelli's Theorem*.
- If g is an integrable function then $|I_1(g)| \leq I(|g|)$ and $|I_2(g)| \leq I(|g|)$.
- If g is an integrable function then $I_1(g) = I(g) = I_2(g)$. This is known as *Fubini's Theorem*.

Most of these are easy. For example, $I_1(g) = I(g) = I_2(g)$ holds when g is a characteristic function because $\mu_1(E) = m_n(E) = \mu_2(E)$. Simple functions are linear combinations of characteristic functions and I_1, I and I_2 are all linear. Non-negative semisimple functions are limits of increasing sequences of simple functions, and all the I's have a version of the Monotone Convergence Theorem.

Tonelli and Fubini (2/4)

Non-negative measurable functions are limits of increasing sequences of non-negative semisimple functions, so we can use the Monotone Convergence Theorem again. We now have Tonelli's Theorem.

> *If g is an integrable function then $|I_1(g)| \leq I(|g|)$ and $|I_2(g)| \leq I(|g|)$.*

This one follows from Tonelli's Theorem, monotonicity and linearity. More precisely, Tonelli gives

$$I_j(|g|) = I(|g|).$$

$-|g| \leq g \leq |g|$ so monotonicity gives

$$I_j(-|g|) \leq I_j(g) \leq I_j(|g|).$$

Linearity gives $I_j(-|g|) = -I_j(|g|)$. Combining these gives

$$-I(|g|) \leq I_j(g) \leq I(|g|).$$

This is just the same as $|I_j(g)| \leq I(|g|)$ though.

Tonelli and Fubini (3/4)

You can prove the general version of Fubini by the same strategy as we used for the First Fundamental Theorem of Calculus (Lebesgue Differentiation Theorem): Every integrable function is the sum of a nice function and an (arbitrarily) function. There's more than one way to define "nice" and "small" but here we use the same one as for the FTC/LDT: "nice" means compactly supported continuous and "small" means L^1 seminorm less than ϵ. The proposition is

> *Suppose* $(X, \mathcal{T})$ *is a locally compact Hausdorff space and* μ *is a Radon measure on* X*. If* $g\colon X \to \mathbf{R}$ *is integrable and* $\epsilon > 0$ *then there is a compactly supported continuous function* $f\colon X \to \mathbf{R}$ *such that*
>
> $$\int_{x \in \mathrm{R}} |f(x) - g(x)| \, d\mu(x) < \epsilon.$$

We apply this with $X = \mathbf{R}^n$ and $\mu = m_n$. In the notation from before, the inequality above is $I(|f - g|) < \epsilon$.

Tonelli and Fubini (4/4)

We have $I(|f-g|) < \epsilon$. From two slides ago, and the linearity of I_j, we have

$$|I_j(f) - I_j(g)| = |I_j(f-g)| \leq I(|f-g|) < \epsilon.$$

We also have

$$|I(f) - I(g)| = |I(f-g)| \leq I(|f-g|) < \epsilon$$

by the linearity and monotonicity of I. By the triangle inequality,

$$|I_j(g) - I(g)| \leq |I_j(g) - I_j(f)| + |I_j(f) - I(f)| + |I(f) - I(g)|.$$

f is compactly supported and continuous so $I_j(f) = I(f)$ and the middle term is zero. The other two are less than ϵ. So $|I_j(g) - I(g)| < 2\epsilon$. This holds for all $\epsilon > 0$, so $I_j(g) = I(g)$. That's Fubini's Theorem.