MAU22200 Lecture 62

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Lebesgue Measure (1/2)

We hope to get Lebesgue measure in $\mathbf{R}^n$ from the Riesz Representation Theorem, just as in $\mathbf{R}$. For this we need the following proposition:

> *Every compactly supported continuous function on $\mathbf{R}^n$ is integrable with respect to Jordan content.*

The proof is straightforward, and mimics the one in $\mathbf{R}$,

- g is supported in some hypercube $[-M, M]^n$.
- g is uniformly continuous there so there is a $\delta > 0$ such that g varies by at most $\frac{\epsilon}{(2M)^n}$ in any ball of radius δ.
- Choose $k > \frac{\sqrt{n}}{\delta}$ and divide $[-M, M]^n$ into $(2Mk)^n$ hypercubes of side length $1/k$. g varies by at most $\frac{\epsilon}{(2M)^n}$ in each. Toss in the complement of $[-M, M]^n$ to get a partition of $\mathbf{R}^n$. g is identically zero there. g can be approximated from above and below by simple functions whose integrals are within ϵ of each other.

Lebesgue Measure (2/2)

$$I(g) = \int_{\mathbf{x}\in\mathbb{R}^n} g(\mathbf{x})\, d\mu_J(\mathbf{x})$$

is therefore well defined. It is linear and has the positivity property that $g \geq 0$ implies $I(g) \geq 0$ The Riesz Representation Theorem gives a Borel measure μ_B such that

$$I(g) = \int_{\mathbf{x}\in\mathbb{R}^n} g(\mathbf{x})\, d\mu_B(\mathbf{x}).$$

We can then complete $(X, \mathcal{B}_B, \mu_B)$ to $(X, \mathcal{B}_L, \mu_L)$. $\mathcal{B}_L$ is called the Lebesgue σ-algebra, its elements are called *Lebesgue measurable sets* μ_L is called *Lebesgue measure*. It's usually denoted m, or m_n if we need to specify the dimension.

Fubini's Theorem for nice functions (1/2)

Suppose g is a compactly supported continuous function on $\mathbf{R}^n$ where $n = n' + n''$. Then $I_1(g) = I(g) = I_2(g)$, where

$$I_1(g) = \int_{\mathbf{x}' \in \mathbf{R}^{n'}} \int_{\mathbf{x}'' \in \mathbf{R}^{n''}} g(\mathbf{x}', \mathbf{x}'')\, d\mu_{n''}(\mathbf{x}'')\, d\mu_{n'}(\mathbf{x}'),$$
$$I_2(g) = \int_{\mathbf{x}'' \in \mathbf{R}^{n''}} \int_{\mathbf{x}' \in \mathbf{R}^{n'}} g(\mathbf{x}', \mathbf{x}'')\, d\mu_{n'}(\mathbf{x}')\, d\mu_{n''}(\mathbf{x}'')$$

The integrals here can be interpreted as integrals with respect to the measure m, the measure μ_B or the content μ_J, since they're all equal for compactly supported continuous functions. We'll use μ_J for the proof, but we'll use the I_j notation later to mean m_n for g which need not be compactly supported and continuous.

Fubini's Theorem for nice functions (2/2)

The proof follows the construction for the proof that compactly supported continuous functions are integrable. If g is simple and corresponds to the partition into hypercubes then $I_1(g) = I(g) = I_2(g)$. This follows from our formula for integrals of simple functions, and the fact the we can rearrange the order of finite sums. In general g isn't simple, but we can find simple f and h with $f \leq g \leq h$ and

$$\int_{\mathrm{x} \in \mathrm{R}^n} h(\mathbf{x})\, d\mu_J(\mathbf{x}) \leq \int_{\mathrm{x} \in \mathrm{R}^n} f(\mathbf{x})\, d\mu_J(\mathbf{x}) + \epsilon$$

It follows that $I_1(g)$, $I(g)$, and $I_2(g)$ are within ϵ of each other. This is true for all $\epsilon > 0$, and g doesn't depend on ϵ, so $I_1(g)$, $I(g)$, and $I_2(g)$ are all equal.

Fubini's Theorem for general functions (1/5)

The real Fubini's Theorem requires only that g is integrable, not compactly supported and continuous. Of course compactly supported continuous functions are integrable, so the version we now have is a special case of the one we want. The general case is easiest to prove in stages.

Most of the time we work with sets rather than functions. Let $\mu_j(E) = I_j(\chi_E)$. Once we have Fubini's Theorem it will follow that $\mu_1(E) = m_n(E) = \mu_2(E)$, but we don't know this initially. Initially we don't even know μ_j is a measure. We'll have to prove its properties by hand.

Fubini's Theorem for general functions (2/5)

The following properties of I_j follow from its definition as the integral of an integral of a characteristic function:

- I_j is linear.
- I_j is monotone, in the sense that if $f \leq g$ then $I_j(f) \leq I_j(g)$.
- There is a version of the Monotone Convergence Theorem for I_j. To get it we just apply the usual Monotone Convergence Theorem to each of the integrals in the definition of I_j.
- There is a version of the Dominated Convergence Theorem for I_j. We just apply the usual Dominated Convergence Theorem to each of the integrals in the definition of I_j.

Fubini's Theorem for general functions (3/5)

The following properties of μ_j follow from those of I_j:

- μ_j is monotone, i.e. $E \subseteq F$ implies $\mu_j(E) \leq \mu_j(F)$.
- μ_j is well behaved with respect to increasing sequences of sets, i.e. if $E_0 \subseteq E_1 \subseteq \cdots$ then

$$\mu_j\left(\bigcup_{k=0}^{\infty} E_k\right) = \lim_{k\to\infty} \mu_j(E_k)$$

  To get this we apply the usual Monotone Convergence Theorem to the sequence of characteristic functions.
- μ_j is well behaved with respect to decreasing sequences of sets, i.e. if $E_0 \supseteq E_1 \supseteq \cdots$ and $\mu_j(E_0) < +\infty$ then

$$\mu_j\left(\bigcap_{k=0}^{\infty} E_k\right) = \lim_{k\to\infty} \mu_j(E_k)$$

  We apply the Dominated Convergence Theorem to the sequence of characteristic functions. The hypothesis that $\mu_j(E_0) < +\infty$ is needed to make this work.

Fubini's Theorem for general functions (4/5)

Now that μ_1 and μ_2 have the properties we expect from a measure we gradually show that they are in fact m_n.

- If K is compact then $\mu_j(K) \leq m_n(K)$.
- If U is open then $\mu_j(U) \geq m_n(U)$.
- If K is compact then $\mu_j(K) = m_n(K)$.
- If U is open then $\mu_j(U) = m_n(U)$.
- If E is Borel then $\mu_j(E) = m_n(E)$.
- If E is Lebesgue measurable then $\mu_j(E) = m_n(E)$.

In spirit, though not in detail, this is like how we proved the Riesz Representation Theorem, gradually weakening hypotheses and strengthening conclusions. I'll sketch the proof of these properties in turn.

Fubini's Theorem for general functions (5/5)

If K is compact then $\mu(K) \leq m_n(K)$. To prove this, use the fact that m_n is a Radon measure, Urysohn's Lemma, and Fubini's Theorem for compactly supported continuous functions.

In more detail, $m_n(K) < +\infty$. We can find a λ such that $m_n(K) < \lambda < +\infty$. Then we can find an open superset U of K such that $m_n(U) < \lambda$. Then can find a compactly supported continuous $h\colon \mathbf{R}^n \to [0, 1]$ such that $h(\mathbf{x}) = 1$ for $\mathbf{x} \in K$ and $h(\mathbf{x}) = 0$ for $\mathbf{x} \notin U$. So $\chi_K \leq h \leq \chi_U$. I_j and I are monotone so $\mu_j(K) = I_j(\chi_K) \leq I_j(h)$ and $I(h) \leq I(\chi_U) = m_n(U)$. h is compactly supported and continuous so $I_j(h) = I(h)$. Therefore $\mu_j(K) \leq m_n(U) < \lambda$. This holds for all $\lambda > m_n(K)$ so $\mu_j(K) \leq m_n(K)$.

The proof that $\mu_j(U) \geq m_n(U)$ if U is open is very similar. It's not quite identical because we don't, for example, have $m_n(U) < +\infty$.

I'll sketch the proof of the remaining properties next time.