MAU22200 Lecture 61

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Bounded semilinear sets and complexes (1/5)

To get a content on $\mathbf{R}^n$ we use two basic geometric facts. The first is

> *Suppose E is a bounded semilinear set. Then there is a simplicial complex $\mathcal{C}$ and a $\mathcal{A} \in \wp(\mathcal{C})$ such that $E = \bigcup_{F \in \mathcal{A}} F^\diamond$.*

Here's a sketch of the proof:

- The Boolean algebra generated by a set $\mathcal{A}$ of sets consists of the finite unions of finite intersections of elements of $\mathcal{A}$ or complements of elements of $\mathcal{A}$.
- The semilinear algebra is generated by open halfspaces therefore consists of the finite unions of finite intersections of open or closed halfspaces.
- Those finite intersections of open or closed halfspaces which are closed and bounded are convex polytopes. We can worry about the non-compact ones later.

Bounded semilinear sets and complexes (2/5)

- This union of convex polytopes isn't necessarily a complex because it might not satisfy the condition that any two intersect in a face, but we can make it into a complex though by subdividing.
- Any complex can be refined to a simplicial complex, as discussed last week.
- We don't need to worry about unbounded pieces because if E is a union of sets at least one of which is unbounded then E is unbounded.
- We do need to worry about pieces which aren't closed, but not very much. We apply the argument above with all the open halfspaces replaced by their closures. The argument above shows that $\overline{E}$ is the underlying space of a simplicial complex $\mathcal{C}$. Any bits which were in $\overline{E}$ but not in E are then removed to get our $\mathcal{A}$.

Bounded semilinear sets and complexes (3/5)

The characterisation of Boolean algebras generated by a set is

> *Suppose X is a set $\mathcal{A} \in \wp(\wp(X))$ and $\mathcal{B}$ is the Boolean algebra generated by $\mathcal{A}$. Let $\mathcal{S}_1$ be the set of E such that $E \in \mathcal{A}$ or $X \setminus E \in \mathcal{A}$. Let $\mathcal{S}_2$ be the set of intersections of finitely many elements of $\mathcal{S}_1$. Let $\mathcal{S}_3$ be the set of unions of finitely many elements of $\mathcal{S}_2$. Then $\mathcal{B} = \mathcal{S}_3$.*

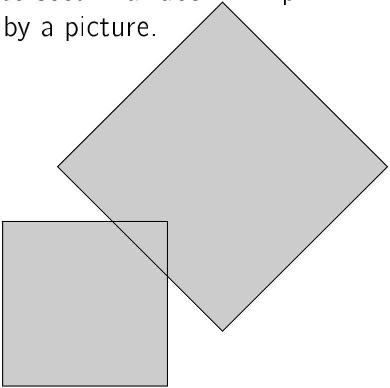
This is analogous to a result we proved for topologies last semester, except there we have arbitrary unions and don't have complements.

The corresponding result for σ-algebras is not true! In particular, not every Borel set is a countable union of countable intersections of open and closed sets.

The proof is in two parts. The easy part is to show that each of $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are subsets of $\mathcal{B}$. The harder part is to show that $\mathcal{S}_3$ is a Boolean algebra. Since $\mathcal{B}$ is the smallest Boolean algebra containing $\mathcal{A}$ it then follows that $\mathcal{S}_3 = \mathcal{B}$.

Bounded semilinear sets and complexes (4/5)

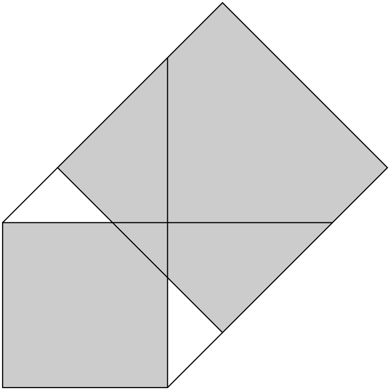
As mentioned previously, the union of convex polytopes isn't necessarily a complex because it might not satisfy the condition that any two intersect in a face. The problem and solution are best illustrated by a picture.



This is the union of two convex polytopes, which happen to be squares. Their intersection isn't a face. Note that splitting it into three doesn't help, since the new pieces aren't all convex.

Bounded semilinear sets and complexes (5/5)

The solution is to extend all the lines in the figure.



The lines extend to infinity, of course, but I've only shown the relevant line segments. Now we've our set into five convex polytopes. This time it is a complex.

In higher dimensions you use hyperplanes rather than lines.

Invariance of volume under refinement (1/5)

Our second basic geometric fact is

Suppose $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ are simplicial complexes, with $\mathcal{C}$ being a refinement of both $\mathcal{C}'$ and $\mathcal{C}''$. For any n dimensional simplex E define $V(E) = \frac{1}{n!}|\det(A_E)|$ where A_E is

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ w_{E,1,0} & w_{E,1,1} & \cdots & w_{E,1,n-1} & w_{E,1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{E,n,0} & w_{E,n,1} & \cdots & w_{E,n,n-1} & w_{E,n,n} \end{bmatrix},$$

$w_{E,i,j}$ is i'th coordinate of $\mathbf{w}_{E,j}$, and the vertices of E are $\mathbf{w}_{E,0}$, $\mathbf{w}_{E,1}$, ..., $\mathbf{w}_{E,n}$. Then

$$\sum_{E \in \mathcal{C}'_n} V(E) = \sum_{E \in \mathcal{C}_n} V(E) = \sum_{E \in \mathcal{C}''_n} V(E)$$

where $\mathcal{C}_n$, $\mathcal{C}'_n$ and $\mathcal{C}''_n$ are the sets of simplices of dimension n in $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$, respectively.

Invariance of volume under refinement (2/5)

It's enough to prove this when you're refining a single simplex. You can then just add over all of the simplices.

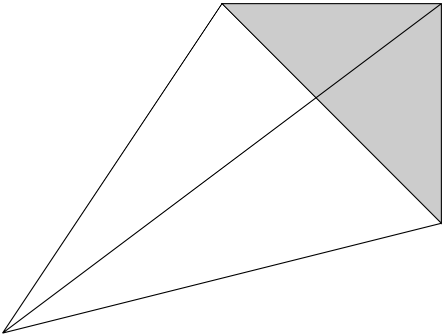
Roughly, if you split a simplex into a finite number of (nearly) disjoint simplices then the volume of the large simplex is the sum of the volume of the small ones.

This is a restatement, not a proof, but it gives the geometric intuition.

The proof is an induction on the dimension. Again, pictures may be helpful.

Invariance of volume under refinement (3/5)

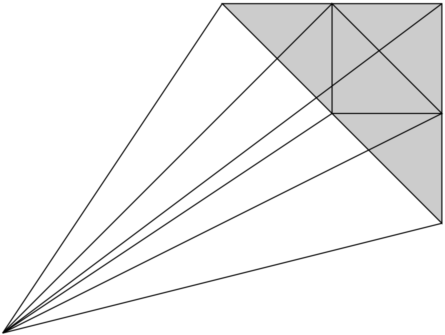
We can write the area of the shaded triangle as a sum of the areas of the three other triangles, with a common vertex.



In the picture two of the triangles occur with a plus sign and one with a minus sign. With a different choice of external vertex that could be different.

Invariance of volume under refinement (4/5)

How does this construction interact with refinement?



There are a lot of triangles! Whenever we have an *internal* edge in the grey triangle it contributes to two triangles, with opposite sides. The *external* edges contribute to only one. So we only need to keep track of the triangles with an edge on one of the edges of the grey triangle. Comparing triangles with the same vertex and opposite edges along the same line, their area is proportional to the length of that opposite edge.

Invariance of volume under refinement (5/5)

We already know that the length of a large edge is the sum of the lengths of the smaller edges into which it's divided. We can now use this to get the same statement for triangles. You can then use that to get the same thing for tetrahedra, etc. You probably don't want to try to draw the pictures though.
The notes have a (mostly) algebraic version of this argument, with one significant difference: I've drawn things with additional vertex external. That makes things easier to draw, but it makes the argument more complicated. If we choose one of the vertices of the original triangle then we only have to worry about triangles connecting that vertex to the opposite edge.

Assembling the pieces

The first of our basic geometric facts allows us to write bounded semilinear sets as unions of simplices, whose volumes we “know” and can add.

The second of our basic geometric facts tells us that the we get the same sum no matter how we split our bounded semilinear set. So defining the content of a bounded semilinear set to be this sum is meaningful.

What about unbounded semilinear sets? We define their content to be infinite. What we get is a content on the semilinear algebra. For $n = 1$ it's the content on $\mathcal{I}$ we defined–much more simply–earlier. For $n > 1$ it's the appropriate generalisation.

For $n > 1$, just as for $n = 1$, we get the Jordan sets and Jordan algebra by completion. You can define the Riemann (or Riemann-Jordan) integral using either the semilinear algebra or the Jordan one, since integrals are insenstive to refinement of the content space.