MAU22200 Lecture 60

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Convex geometry (1/6)

Last time we had the problem that the intersection of triangles generally isn't a triangle. It could be empty a point, a line segment, a triangle, a quadrilateral, a pentagon or a hexagon. One thing we can be sure of is that it's convex.
For tetrahedra there are many possibilities, but again they're all convex. In general, simplices are convex and intersections of convex sets are convex. A lot of the geometry we need for understanding semilinear sets is convex geometry.
Hopefully you already have some intuition about convex sets. There's a chapter in the notes on affine and convex geometry to fill this in somewhat and add some details which are probably unfamiliar. I've mostly not bothered with proofs in that chapter.

Convex geometry (2/6)

An *affine* space is a subset of $\mathbf{R}^n$ which, if it contains $\mathbf{x}$ and $\mathbf{y}$, also contains the line $(1-t)\mathbf{x}+t\mathbf{y}$ for $t \in \mathbf{R}$.

A *convex* subset of $\mathbf{R}^n$ is one which, if it contains $\mathbf{x}$ and $\mathbf{y}$, also contains the line segment $(1-t)\mathbf{x}+t\mathbf{y}$ for $t \in [0,1]$.

Affine sets are convex, but most convex sets aren't affine. Affine sets are just translates of (linear) subspaces.

The *affine span* of a set is the smallest affine space containing it.

The *convex hull* of a set is the smallest convex set containing it.

The *dimension* of a convex set is defined to be the dimension of its affine span. This definition behaves as expected, but it wouldn't if we didn't restrict it to convex sets.

Everything I'm going to say would work with an arbitrary finite dimensional normed vector space in place $\mathbf{R}^n$, and some of it would work with just a normed vector space.

Convex geometry (3/6)

It's useful to talk about the *relative interior* of a convex set. For example, the relative interior of the line segment $(1-t)\mathbf{x}+t\mathbf{y}$ for $t \in [0,1]$ is the set of $(1-t)\mathbf{x}+t\mathbf{y}$ for $t \in (0,1)$ except if $\mathbf{x}=\mathbf{y}$, in which case it's $\mathbf{x}$. In general, the relative interior of a set is its interior when considered as subset of its affine span, with the subspace topology. We say it's *relatively open* if it is equal to its relative interior.

We don't introduce a notion of relative closure though.

If C is non-empty convex set then the relative interior of C is a non-empty relatively open convex set contained in C and the closure of C is a non-empty closed convex set containing C. This is geometrically intuitive, but not so easy to prove.

Convex geometry (4/6)

Examples of convex sets include the open halfspace $\sum_{i=1}^{n} a_i x_i + b > 0$ and the closed halfspace $\sum_{i=1}^{n} a_i x_i + b \geq 0$. Here $a_1, \ldots, a_n$ are assumed not all to be zero. The open halfspace is the relative interior of the closed halfspace and the closed halfspace is the closure of the open halfspace.

$\mathbf{x}_0, \ldots, \mathbf{x}_k$ are said to be *affinely* independent if their affine span is of dimension k. In that case the simplex $\sum_{i=0}^{k} t_i \mathbf{x}_i$ with $t_0, t_1, \ldots, t_k \geq 0$ and $\sum_{i=0}^{k} t_i = 1$ is a closed convex set of dimension k. Its relative interior is the set of points $\sum_{i=0}^{k} t_i \mathbf{x}_i$ with $t_0, t_1, \ldots, t_k > 0$ and $\sum_{i=0}^{k} t_i = 1$. The closure of the latter is the former.

In general the closure of a convex set is equal to the closure of its relative interior and the relative interior of a convex set is equal to the relative interior of its closure. In this respect convex sets are better behaved topologically than sets in general.

Convex geometry (5/6)

A *convex polytope* is the intersection of finitely many closed halfspaces. We want to define a *complex* as a finite collection of bounded convex polytopes, joined together at faces. To do this, we need a definition of faces. It turns out to be simplest to do this for general convex sets, not specifically for bounded convex polytopes.

F is said to be a *face* of C if $\mathbf{x}, \mathbf{y} \in F$ whenever $\mathbf{x}, \mathbf{y} \in C$, $(1-t)\mathbf{x} + t\mathbf{y} \in F$ and $t \in (0,1)$.

This is not quite the terminology you're used to from polyhedra in $\mathbf{R}^3$. A k-dimensional bounded convex polytope has faces of dimensions -1 through k, not just dimension $k-1$. Think of faces as corresponding to vertices, edges, "faces", etc.

Convex geometry (6/6)

The definition of a face isn't intuitive, but it does have the expected properties.

- Faces of convex sets are convex.
- Faces of closed convex sets are closed convex sets.
- If E is a face of F and F is a face of C then E is a face of C.
- If E and F are faces of C then $E \cap F$ is a face of C.
- If E is a face of C and $E \neq C$ then the dimension of E is less than the dimension of C.
- The set where a linear function on a convex set takes its maximum is a face of that set.
- If the relative interiors of two faces have a point in common then they are the same face.
- A convex polytope has only finitely many faces.

Complexes (1/3)

A finite set $\mathcal{C}$ of compact convex polytopes is called a *complex* if $F \in \mathcal{C}$ whenever $E \in \mathcal{C}$ and F is a face of E and if $E \cap F$ is a face of both E and F whenever $E \in \mathcal{C}$ and $F \in \mathcal{C}$.

The underlying set of $\mathcal{C}$ is $\bigcup_{E \in \mathcal{C}} E$. Note a complex is a finite set of subsets of $\mathbf{R}^n$ while its underlying set is a (generally infinite) subset of $\mathbf{R}^n$. Also, different complexes can have the same underlying set. Starting from a subset $S \subseteq \mathbf{R}^n$ you can think of a complex $\mathcal{C}$ such that S is the underlying set of $\mathcal{C}$ as a decomposition of S. It's not quite a partition because faces intersect, but the relative interiors of the non-empty elements of $\mathcal{C}$ are a partition of S.

The *dimension* of a complex is the maximum of the dimensions of its elements. The *mesh* of a complex is the maximum of the diameters of its elements.

A complex is called *simplicial* if all of its elements are simplices,

Complexes (2/3)

A *subcomplex* of a complex is a subset which is also a complex. Not every subset is a subcomplex. If we take a subset which contains a compact convex polytope but not one of its faces then it won't be a subcomplex.

A complex $\mathcal{C}'$ is said to be a *refinement* of a complex $\mathcal{C}$ if they have the same underlying set and for every $E' \in \mathcal{C}'$ there is an $E \in \mathcal{C}$ such that $E' \subseteq E$.

We can get a common refinement of two complexes $\mathcal{C}_1$ and $\mathcal{C}_2$ with the same underlying set by taking sets of the form $E_1 \cap E_2$ where $E_1 \in \mathcal{C}_1$ and $E_2 \in \mathcal{C}_2$. This complex generally isn't simplicial, even if $\mathcal{C}_1$ and $\mathcal{C}_2$ were.

Any complex has a simplicial refinement though. To get such a refinement, choose a $\varphi(E)$ in the relative interior of E for each non-empty $E \in \mathcal{C}$. We take the elements of $\mathcal{C}'$ to be the simplices with vertices $\varphi(E_0), \ldots, \varphi(E_k)$ for each strictly increasing chain $E_0 \subset E_1 \subset \cdots \subset E_k$ of non-empty elements of $\mathcal{C}$

For an illustration of both procedures, look at the pictures from the end of last lecture.

Complexes (3/3)

To define a content on the semilinear algebra we need two basic facts.

> *If E is a bounded semilinear set then there is a simplicial complex $\mathcal{C}$ and a subset $\mathcal{A} \subseteq C$ such that E is the union of the relative interiors of the elements of $\mathcal{A}$. In fact for each $\delta > 0$ there is one with mesh less than δ.*

and

> *The sum of the contents of the n-dimensional simplices in the $\mathcal{A}$ depends only on E, not on $\mathcal{A}$ and $\mathcal{C}$.*