MAU22200 Lecture 59

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How we developed Lebesgue integration in **R** (1/2)

1. We started from a collection of sets for which we know how their measure should be defined: the intervals.
2. We extended that collection to the Boolean algebra it generates, $\mathcal{I}$. We can write each element of $\mathcal{I}$ as a union of intervals, but generally in more than one way.
3. We extended the length to a content on $\mathcal{I}$. We can do this by summing the lengths of the integrals, but some argument is needed to show that this is independent of *how* we write our set as a union of intervals. This is already adequate for integrating compactly supported continuous functions.
4. We complete our Boolean algebra and content to get the Jordan algebra and Jordan content.
5. We use the Riesz Representation Theorem to get a Borel measure.
6. We complete again to get Lebesgue measure.

How we developed Lebesgue integration in **R** (2/2)

- Step 1 (defining the length of an interval was trivial).
- Step 2 (understanding the Boolean algebra generated by the intervals) wasn't much harder.
- Step 3 (extending length by summation) required a trick to show that the content is well defined, i.e. independent of *how* we write our set as a union of intervals. The fact that compactly supported continuous functions are integrable follows from uniform continuity and the fact that we can partition an interval into arbritrarily small subintervals.
- Step 4 (completion of the content) was long and messy.
- Step 5 (the proof of the Riesz Representation Theorem) was even longer and messier.
- Step 6 (completion of the measure) was easy, because we already did all the work in Step 4.

How to develop Lebesgue integration in $\mathbf{R}^n$

We can follow essentially the same procedure in higher dimensions.

The steps which were hard, Steps 4-6, are now easy, because we did them in enough generality that we don't need to repeat them. Completion just requires a content/measure space and the Riesz Representation Theorem just requires a locally compact σ-compact Hausdorff topological space. Almost all the structure of $\mathbf{R}^n$ is irrelevant.

Unfortunately, the steps which were easy are now hard, especially Steps 2 and 3. This is essentially a problem in Geometry, and Geometry is just a lot more complicated in higher dimensions than in one dimension.

Step 1 (1/4)

For Step 1 we need some collection of sets for which we know what the measure should be. In $\mathbf{R}^2$ we could take triangles or rectangles. In $\mathbf{R}^3$ we could take tetrahedra or rectangular solids. In $\mathbf{R}^n$ we could take simplices or boxes. A simplex is the higher dimensional generalisation of an interval, triangle or tetrahedron. A box is the higher dimensional generalisation of an interval, rectangle or rectangular solid. The term "simplex" is standard. The term "box" less so.

A box is a Cartesian product of intervals. Its content is the product of their lengths. A simplex is the set of points of the form $\sum_{i=0}^{n} t_i \mathbf{x}_i$ for some $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$, where $t_0, t_1, \ldots, t_n \geq 0$ and $\sum_{i=0}^{n} t_i = 1$.

Step 1 (2/4)

The content of the simplex is

$$\frac{1}{n!}\left|\det\left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_{0,1} & x_{1,1} & \cdots & x_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{0,n} & x_{1,n} & \cdots & x_{n,n} \end{bmatrix}\right)\right|$$

where $x_{i,j}$ is the i'th coordinate of $\mathbf{x}_j$. If $n = 1$ the simplex is the set of points $t_0 x_0 + t_1 x_1$ where $t_0, t_1 \geq 0$ and $t_0 + t_1 = 1$. I haven't used indices for the coordinates because there's only one coordinate! In other words, it's the interval $[\min(t_0, t_1), \max(t_0, t_1)]$. The content (length) of this simplex (interval) is

$$\left|\det\left(\begin{bmatrix} 1 & 1 \\ x_0 & x_1 \end{bmatrix}\right)\right| = |x_1 - x_0|,$$

as expected.

Step 1 (3/4)

If $n = 2$ the simplex is the set of points $t_0(x_0, y_0) + t_1(x_1, y_1) + t_2(x_2, y_2)$, where $t_0, t_1, t_2 \geq 0$ and $t_0 + t_1 + t_2$. In other words, it's the triangle with vertices (x_0, y_0), (x_1, y_1) and (x_2, y_2). As is traditional, I've used x and y as coordinates in the plane rather than x_1 and x_2. This time there's no preferred ordering of the vertices. The content (area) of this simplex (triangle) is

$$\frac{1}{2}\left|\det\left(\begin{bmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{bmatrix}\right)\right| = \frac{|x_1 y_2 + x_2 y_0 + x_0 y_1 - x_2 y_1 - x_0 y_2 - x_1 y_0|}{2}.$$

This is indeed the correct formula for the area of the triangle with vertices (x_0, y_0), (x_1, y_1) and (x_2, y_2).

Step 1 (4/4)

I'll spare you the case $n = 3$, but the general formula does indeed give the correct value for the volume of the tetrahedron with vertices (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). Do we want to choose simplices or boxes? The box version looks simpler but it has two disadvantages. Aesthetically, it seems wrong to define area, volume, etc. of general sets in $\mathbf{R}^2$, $\mathbf{R}^3$, etc. in a way which strongly depends on the choice of axes. Practically, it becomes surprisingly hard to show that congruent sets have equal area, volume, etc. The simplex version has a definition which appears to depend on the choice of axes, but it's an easy Linear Algebra calculation to see that it doesn't. If we choose simplices we can use the fact that any rotation or reflection of a simplex is another simplex. The rotation or reflection of a box isn't generally a box.

Step 2 (1/3)

We now need to understand the Boolean algebra generated by the simplices. That Boolean algebra is a bit messy. It's convenient to use a slightly larger Boolean algebra, generated by the open halfspaces. An open half space is the set where a linear, not necessarily homogeneous, function is positive. This Boolean algebra is called the semilinear algebra. Its elements are called semilinear sets. In $n = 1$ the semilinear algebra is $\mathcal{I}$ The Boolean algebra generated by the simplices, i.e. finite closed intervals, would be smaller, since it doesn't contain $[0, +\infty)$.

Step 2 (2/3)

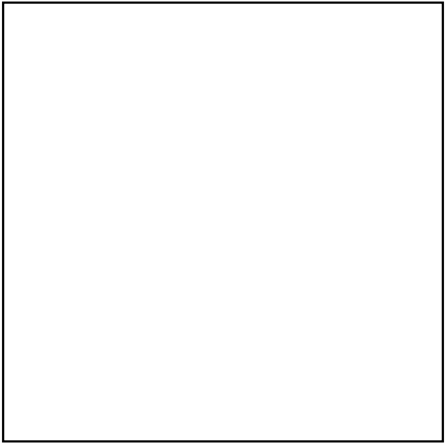
Every bounded element of $\mathcal{I}$ can be partitioned by points and open intervals. There's a minimal such partition.
Every bounded semilinear set in $\mathbf{R}^2$ can be partitioned by points, the interiors of line segments, and the interiors of triangles. For example, the closed unit square is the union of four points, five interiors of line segments, and the interiors of two triangles. One triangle has vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$. The other has vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. The line segments are the sides of these triangles and the points are their vertices. This time there's no minimal partition. The triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$ and the one with vertices $(0, 1)$, $(1, 0)$ and $(1, 1)$ would also work, but there's no partition of which both are refinements. What we're doing here is called *triangulation* in $\mathbf{R}^2$ and *simplicial decomposition* in $\mathbf{R}^n$.

Step 2 (3/3)

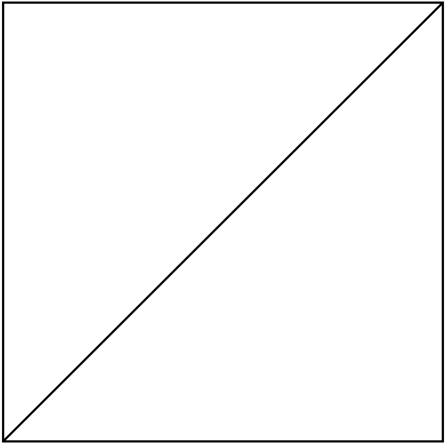
There are better and worse ways to triangulate a semilinear set in $\mathbf{R}^2$. We could split the unit square into the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$, one with vertices $(0,1)$, $(1/2,1/2)$ and $(1,1)$ and one with vertices $(1,0)$, $(1/2,1/2)$ and $(1,1)$. The point $(1/2,1/2)$ is a vertex of two triangles but is an interior point of a side of the other. This isn't the end of the world, but it's awkward and avoidable.

One can define a *simplicial complex*, which corresponds to a good triangulation, i.e. one which avoids complications like the one above. Every bounded semilinear set has a simple description in terms of a simplicial complex.

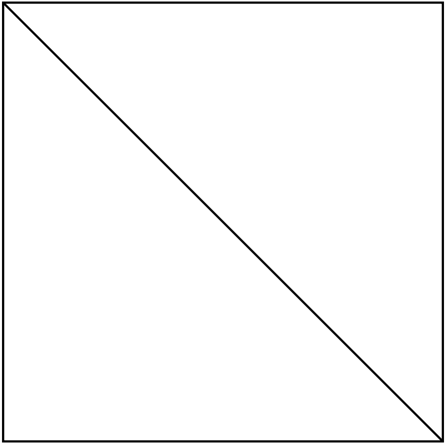
Pictures (1/4)



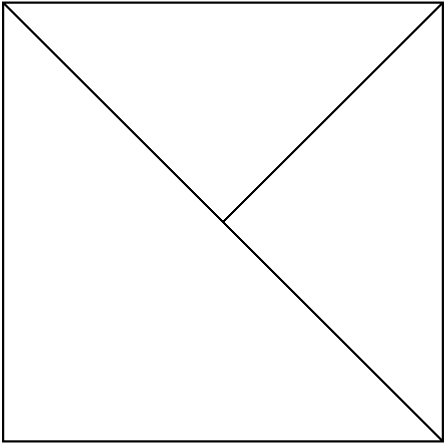
Pictures (2/4)



Pictures (3/4)



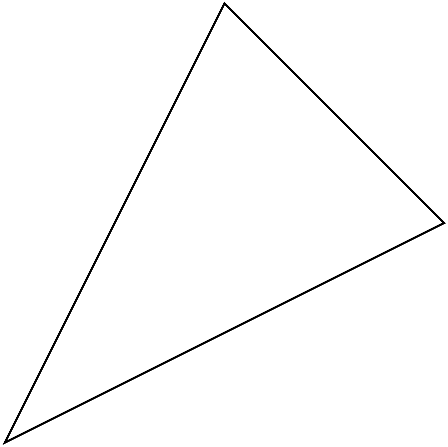
Pictures (4/4)



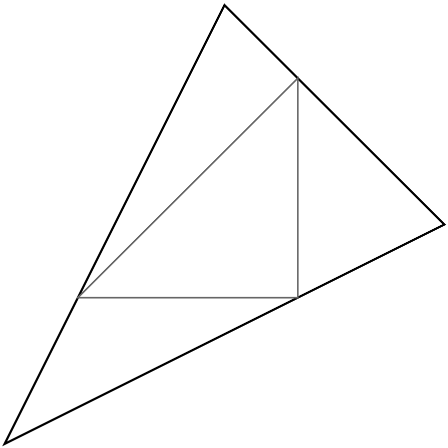
Step 3

We can calculate the area/volume/content of a bounded semilinear set from the simplicial complex. Essentially, we just add the area/volume/contents of the simplices of dimension 2/3/n, which are calculated from the formula given earlier. I said "calculate" rather than "define" because it's not obvious this is a property of the semilinear set and not of the simplicial complex. There are many different simplicial complexes which give the same semilinear set. Does the sum of area/volume/contents of the simplices depend on which one is chosen? Luckily it doesn't, but this requires a proof. We can prove this by introducing a notion of refinement, such that any two such partitions have a common refinement and show that the area/volume/content is the same for a partition and its refinement. There's a catch, though. The intersection of two intervals is an interval but the intersection of triangles/simplices isn't generally a triangle/simplex! So the obvious way of getting a common refinement doesn't work.

More pictures (1/8)



More pictures (2/8)



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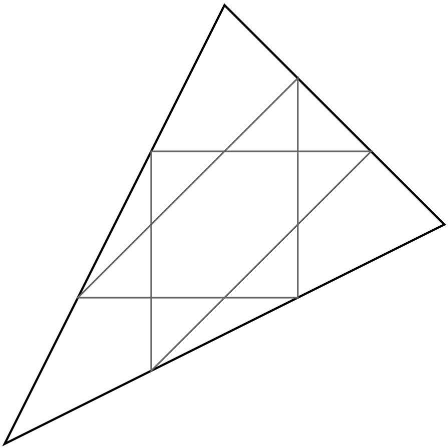
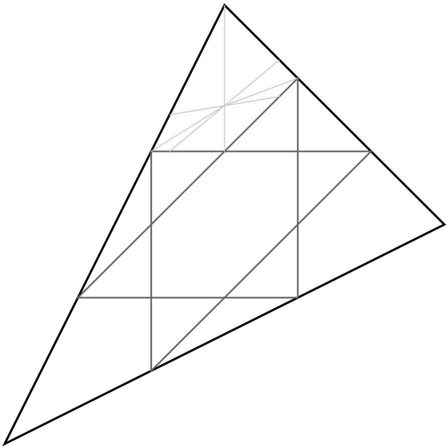


More pictures (4/8)



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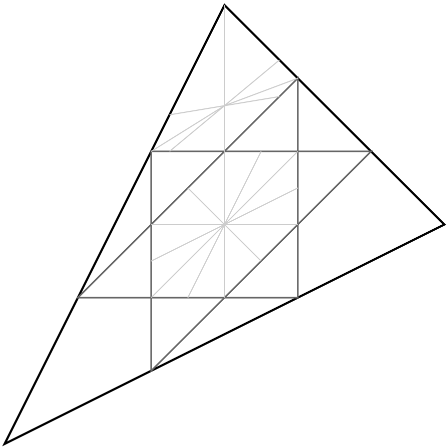

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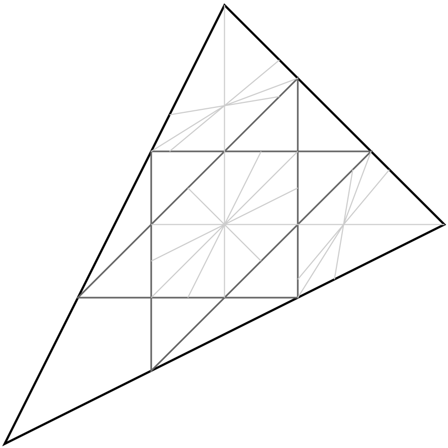

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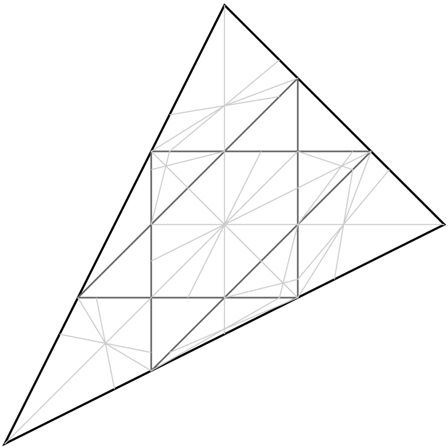

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