MAU22200 Lecture 58

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28 March 2022

Where we are now

We are trying to prove

> *Suppose* $F\colon [a,b] \to \mathbf{R}$ *is Lipschitz continuous. Let* E *be the set of* x *such that* $F'(x)$ *is differentiable at* x*. Then* $m([a,b] \setminus E) = 0$*,* F' *is integrable on* $E \cap [a,b]$ *and*
>
> $$\int_{x \in E \cap [a,b]} F'(x)\, dm(x) = F(b) - F(a).$$

We saw how to prove this from two propositions. I sketched the proof of the first one,

> *Suppose* I *is a non-empty interval and* $F\colon I \to \mathbf{R}$ *is Lipschitz continuous. Then there are Lipschitz continuous functions* G *and* H *such that* G *is monotone increasing,* H *is monotone decreasing and* $F = G + H$*.*

last time.

Dini derivatives (1/9)

Today I'll sketch the proof of the second one,

> *Suppose I is a non-empty interval and $F: I \to \mathbf{R}$ is continuous and monotone. Then F is differentiable at x for almost all $x \in I$.*

Even if you don't know anything about the possible differentiability of a function you can still define the *Dini derivatives*

$$D^+F(x) = \limsup_{h \searrow 0} \frac{F(x+h) - F(x)}{h},$$

$$D_+F(x) = \liminf_{h \searrow 0} \frac{F(x+h) - F(x)}{h},$$

$$D^-F(x) = \limsup_{h \nearrow 0} \frac{F(x+h) - F(x)}{h}$$

$$D_-F(x) = \liminf_{h \nearrow 0} \frac{F(x+h) - F(x)}{h}.$$

Dini derivatives (2/9)

The following are useful properties of the Dini derivatives.

- Suppose F is a function from an open interval in $\mathbf{R}$ to $\mathbf{R}$. Then $D_+F(x) \leq DF^+(x)$ and $D_-F(x) \leq DF^-(x)$. If F is monotone increasing then $0 \leq D_+F(x) \leq DF^+(x)$ and $0 \leq D_-F(x) \leq DF^-(x)$.
- F is differentiable at x if and only if

$$D^+F(x) = D_+F(x) = D^-F(x) = D_-F(x) \in \mathbf{R}.$$

- If I is an interval and $F\colon I \to \mathbf{R}$ is continuous then the Dini derivatives of F are all measurable.

The first two are easy. The last is more complicated. In fact the continuity assumption is not needed, but it makes the proof considerably simpler. The proof illustrates some useful tricks, so I'll give it in full.

Dini derivatives (3/9)

It suffices to prove measurability for one of the four Dini derivatives. The other three can then be obtained by replacing $F(x)$ by $F(-x)$, $-F(x)$ or $-F(-x)$. I'll prove it for DF^+.

$$DF^+(x) = \limsup_{h \to 0} \frac{F(x+h) - F(x)}{h}$$
$$= \inf_{k \in (0,+\infty)} \sup_{h \in (0,k)} \frac{F(x+h) - F(x)}{h}.$$

Infima and suprema of *countable* collections of measurable sets are measurable, but unfortunately $(0, +\infty)$ and $(0, k)$ are uncountable. We need to get around this fact somehow. Let

$$y = \sup_{h \in (0,k) \cap Q} \frac{F(x+h) - F(x)}{h}.$$

Then

$$y \leq \sup_{h \in (0,k)} \frac{F(x+h) - F(x)}{h}$$

because $(0, k) \cap \mathbf{Q}$ is a subset of $(0, k)$.

Dini derivatives (4/9)

Suppose

$$y < \sup_{h \in (0,k)} \frac{F(x+h) - F(x)}{h}.$$

Then there is an $h \in (0, k)$ such that

$$y < \frac{F(x+h) - F(x)}{h}.$$

Another way to say this is that the set

$$\left\{ h \in (0, k) \colon \frac{F(x+h) - F(x)}{h} > y \right\}$$

is non-empty. It's open by the continuity of F and every non-empty subset of the reals contains a rational, so there is an $h \in (0, k) \cap \mathbf{Q}$ such that

$$y < \frac{F(x+h) - F(x)}{h}$$

Dini derivatives (5/9)

$$y = \sup_{h \in (0,k) \cap Q} \frac{F(x+h) - F(x)}{h},$$

$h \in (0, k) \cap \mathbf{Q}$ and

$$y < \frac{F(x+h) - F(x)}{h}.$$

But this is impossssible so the assumption that

$$y < \sup_{h \in (0,k)} \frac{F(x+h) - F(x)}{h}$$

was false and therefore

$$y = \sup_{h \in (0,k)} \frac{F(x+h) - F(x)}{h},$$

i.e.

$$\sup_{h \in (0,k) \cap Q} \frac{F(x+h) - F(x)}{h} = \sup_{h \in (0,k)} \frac{F(x+h) - F(x)}{h}.$$

Dini derivatives (6/9)

$$\sup_{h \in (0,k) \cap Q} \frac{F(x+h) - F(x)}{h} = \sup_{h \in (0,k)} \frac{F(x+h) - F(x)}{h}$$

so

$$DF^+(x) = \inf_{k \in (0,+\infty)} \sup_{h \in (0,k)} \frac{F(x+h) - F(x)}{h}.$$

$$= \inf_{k \in (0,+\infty)} \sup_{h \in (0,k) \cap Q} \frac{F(x+h) - F(x)}{h}.$$

Let

$$z = \inf_{k \in (0,+\infty) \cap Q} \sup_{h \in (0,k) \cap Q} \frac{F(x+h) - F(x)}{h}.$$

$(0, +\infty) \cap \mathbf{Q}$ is a subset of $(0, +\infty)$ so

$$z \geq \inf_{k \in (0,+\infty)} \sup_{h \in (0,k) \cap Q} \frac{F(x+h) - F(x)}{h}$$

Dini derivatives (7/9)

If

$$z > \inf_{k \in (0,+\infty)} \sup_{h \in (0,k) \cap Q} \frac{F(x+h) - F(x)}{h}$$

then there is a $k \in (0 + \infty)$ such that

$$z > \sup_{h \in (0,k) \cap Q} \frac{F(x+h) - F(x)}{h}$$

Every non-empty open subset of the reals contains a rational number so there is a $j \in (0, k) \cap \mathbf{Q}$. Then

$$(0, j) \cap \mathbf{Q} \subseteq (0, k) \cap \mathbf{Q}$$

so

$$\sup_{h \in (0,j) \cap Q} \frac{F(x+h) - F(x)}{h} \leq \sup_{h \in (0,k) \cap Q} \frac{F(x+h) - F(x)}{h}$$

and therefore

$$z > \sup_{h \in (0,j) \cap Q} \frac{F(x+h) - F(x)}{h}.$$

Dini derivatives (8/9)

$$z = \inf_{k \in (0,+\infty) \cap \mathbb{Q}} \sup_{h \in (0,k) \cap \mathbb{Q}} \frac{F(x+h) - F(x)}{h},$$

$j \in (0, k) \cap \mathbf{Q}$, and

$$z > \sup_{h \in (0,j) \cap \mathbb{Q}} \frac{F(x+h) - F(x)}{h}.$$

But this is impossible so

$$z = \inf_{k \in (0,+\infty)} \sup_{h \in (0,k) \cap \mathbb{Q}} \frac{F(x+h) - F(x)}{h}.$$

In other words,

$$D^+ F(x) = \inf_{k \in (0,+\infty) \cap \mathbb{Q}} \sup_{h \in (0,k) \cap \mathbb{Q}} \frac{F(x+h) - F(x)}{h}.$$

Now we can use the fact that infima and suprema of countable collections of measurable functions are measurable.

Dini derivatives (9/9)

Suppose $F\colon [a,b] \to \mathbf{R}$ *is monotone increasing and* $\lambda > 0$*. Then there is a* $C > 0$ *such that*

$$m\left(\left\{x \in [a,b]\colon D^+F(x) \geq \lambda\right\}\right) \leq C\,\frac{F(b)-F(a)}{\lambda}$$

$$m\left(\{x \in [a,b]\colon D_+F(x) \geq \lambda\}\right) \leq C\,\frac{F(b)-F(a)}{\lambda}$$

$$m\left(\left\{x \in [a,b]\colon D^-F(x) \geq \lambda\right\}\right) \leq C\,\frac{F(b)-F(a)}{\lambda}$$

$$m\left(\{x \in [a,b]\colon D_-F(x) \geq \lambda\}\right) \leq C\,\frac{F(b)-F(a)}{\lambda}$$

If F *is continuous then the inequalities above hold with* $C = 1$*.*

The proof is long and I won't give it here. We only need the continuous case in order to prove the Fundamental Theorem of Calculus. This would follow from Hardy-Littlewood if we already had the Second Fundamental Theorem of Calculus.

Differentiability almost everywhere (1/2)

The proof that continuous monotone functions are differentiable almost everywhere is long but the main ideas are:

- $D_+F(x) \leq D^+F(x)$ and $D_-F(x) \leq D^-F(x)$ for all x.
- If we can show that $D^+F(x) \leq D_-F(x)$ and $D^-F(x) \leq D_+F(x)$ for all almost all x then F we can conclude that F is differentiable for almost all x.
- If $D^+F(x) > D_-F(x)$ then there are $p, q \in \mathbf{Q}$ such that

$$D_-F(x) < p < q < D^+F(x).$$

- If we can show that $m(A_{p,q}) = 0$, where

$$A_{p,q} = \left\{x \colon D_-F(x) < p < q < D^+F(x)\right\}$$

then $m\left(\bigcup A_{p,q}\right) = 0$ and $D^+F(x) \leq D_-F(x)$ almost everywhere.

Differentiability almost everywhere (2/2)

- Similarly, if we can show that $m(A_{p,q}) = 0$, where

$$B_{p,q} = \left\{x \colon D_+F(x) < p < q < D^-F(x)\right\}$$

  then $m\left(\bigcup B_{p,q}\right) = 0$ and $D^-F(x) \leq D_+F(x)$ almost everywhere.

- To show that $A_{p,q}$ is null we first show that its average value is bounded by $p/q < 1$ almost everywhere, i.e. that

$$\frac{1}{2h} m(A_{p,q} \cap (x-h, x+h)) \leq \frac{p}{q}.$$

  This follows from the "Hardy-Littlewood" inequalities.

- The Lebesgue Differentiation Theorem says that this average is equal to $\chi_{A_{p,q}}(x)$ for almost all x. In other words, almost all x are not in $A_{p,q}$.