MAU22200 Lecture 57

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The Lebesgue Differentiation Theorem

Last time we saw this variant of the First Fundamental Theorem:

Suppose $f : [a, b] \to \mathbf{R}$ *is integrable. Then*

$$\lim_{h \searrow 0} \frac{1}{h} \int_{x \in [y, y+h]} f(x)\, dm(x) = f(y),$$

$$\lim_{h \searrow 0} \frac{1}{h} \int_{x \in [y-h, h]} f(x)\, dm(x) = f(y),$$

$$\lim_{h \searrow 0} \frac{1}{2h} \int_{x \in [y-h, y+h]} f(x)\, dm(x) = f(y)$$

for almost all $x \in [a, b]$.

The last bit is the one-dimensional version of the Lebesgue Differentiation Theorem: For almost all $\mathbf{x}$,

$$\lim_{r \searrow 0} \frac{1}{m(B(\mathbf{y}, r))} \int_{x \in B(y, r)} f(\mathbf{x})\, dm(\mathbf{x}) = f(\mathbf{y}).$$

I'll sometimes refer to the other parts as the Lebesgue Differentiation Theorem too.

The Second Fundamental Theorem of Calculus (1/5)

Suppose $F\colon [a, b] \to \mathbf{R}$ is Lipschitz continuous. Then F is differentiable at x for almost all $x \in [a, b]$ and

$$\int_{x\in[a,b]} F'(x)\, dm(x) = F(b) - F(a).$$

Suppose that F is differentiable at x for almost all $x \in [a, b]$. We can get

$$\int_{x\in[a,b]} F'(x)\, dm(x) = F(b) - F(a).$$

as follows. If $h_0, h_1, \ldots$ are positive and tend to zero then the functions

$$g_j(x) = \frac{F(x + h_j) - F(x)}{h_j}$$

converge for almost all x and tend to $F'(x)$. If F is Lipschitz with Lipschitz constant K then $|g_j(x)| \leq K$.

The Second Fundamental Theorem of Calculus (2/5)

$$\int_{x\in[a,b]} K \, dm(x) < +\infty.$$

By the Dominated Convergence Theorem

$$\int_{x\in[a,b]} F'(x) \, dm(x) = \lim_{j\to\infty} \int_{x\in[a,b]} g_j(x) \, dm(x).$$

$$\begin{aligned}
\int_{x\in[a,b]} g_j(x) \, dm(x) &= \int_{x\in[a,b]} \frac{F(x+h_j)-F(x)}{h_j} \, dm(x) \\
&= \frac{1}{h_j}\left[\int_{x\in[a,b]} F(x+h_j) \, dm(x) - \int_{x\in[a,b]} F(x) \, dm(x)\right] \\
&= \frac{1}{h_j}\left[\int_{x\in[a+h_j,b+h_j]} F(x) \, dm(x) - \int_{x\in[a,b]} F(x) \, dm(x)\right] \\
&= \frac{1}{h_j}\int_{x\in[b,b+h_j]} F(x) \, dm(x) - \frac{1}{h_j}\int_{x\in[a,a+h_j]} F(x) \, dm(x)
\end{aligned}$$

The Second Fundamental Theorem of Calculus (3/5)

If F is Lipschitz continuous then F is continuous and

$$\lim_{j\to\infty} \frac{1}{h_j} \int_{x\in[a,a+h_j]} F(x)\, dm(x) = F(a)$$

and

$$\lim_{j\to\infty} \frac{1}{h_j} \int_{x\in[b,b+h_j]} F(x)\, dm(x) = F(b)$$

You can use the Riemann integration version of the First Fundamental Theorem of Calculus together with the fact that the Lebesgue integral agrees with the Riemann integral where both are defined. Or you can use Lebesgue version, i.e. the Lebesgue Differentiation Theorem, but you might need to shift a and b slightly and take a limit. Either way,

$$F(b) - F(a) = \int_{x\in[a,b]} F'(x)\, dm(x).$$

The Second Fundamental Theorem of Calculus (4/5)

The problem is to show that every Lipschitz continuous function is differentiable almost everywhere and that the derivative is integrable. We'll get this mostly from two propositions.
Proposition 10.3.1 says

> *Suppose I is a non-empty interval and $F \colon I \to \mathbf{R}$ is Lipschitz continuous. Then there are Lipschitz continuous functions G and H such that G is monotone increasing, H is monotone decreasing and $F = G + H$.*

and Proposition 10.3.8 says

> *Suppose I is a non-empty interval and $F \colon I \to \mathbf{R}$ is continuous and monotone. Then F is differentiable at x for almost all $x \in X$.*

We already know that Lipschitz continuous functions are continuous. Also, the sum of two functions which are differentiable almost everywhere is differentiable almost everywhere.

The Second Fundamental Theorem of Calculus (5/5)

More miscellaneous bits we need are that the derivative of a continuous function is measurable and that the derivatives of a Lipschitz function are bounded everywhere. Bounded measurable functions on a set of finite measure are integrable, so F' is integrable.

The hard part is proving the two propositions from the previous slide. I won't give the full proof of either in lecture, but I will sketch the proofs and describe some useful constructions.

Variation (1/2)

We define the positive variation, negative variations and and total variation of a function in a closed interval by

$$V_+(a,b) = \sup \sum_{j=1}^{n} \max(0, F(x_j) - F(x_{j-1})),$$

$$V_-(a,b) = \inf \sum_{j=1}^{n} \min(0, F(x_j) - F(x_{j-1})),$$

$$V(a,b) = \sup \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|.$$

The suprema and infima are over $x_0, x_1, \ldots, x_n$ are such that

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b.$$

For general functions they could be infinite but for Lipschitz functions they're of absolute value at most $K(b-a)$ where K is a Lipschitz constant for F.

Variation (2/2)

The variations have the following properties:

- $V_+(a,b) \geq 0$, $V_-(a,b) \leq 0$ and $V(a,b) \geq 0$.
- if $a \leq b \leq c$ then

$$\begin{aligned} V_+(a,c) &= V_+(a,b) + V_+(b,c), \\ V_-(a,c) &= V_-(a,b) + V_-(b,c), \\ V(a,c) &= V(a,b) + V(b,c). \end{aligned}$$

- $V_+(a,c) \geq V_+(a,b)$ and $V_+(a,c) \geq V_+(b,c)$. So $V_+(a,c)$ is monotone increasing as a function of c and monotone decreasing as a function of a. The same is true for V, while $V_-(a,c)$ is monotone decreasing as a function of c and monotone increasing as a function of a.
- $V_+(a,b) - V_-(a,b) = V(a,b)$.
- $V_+(a,b) + V_-(a,b) = F(b) - F(a)$.

Decomposition of Lipschitz Functions

One of the two propositions we wanted was

> *Suppose I is a non-empty interval and $F\colon I \to \mathbf{R}$ is Lipschitz continuous. Then there are Lipschitz continuous functions G and H such that G is monotone increasing, H is monotone decreasing and $F = G + H$.*

From the properties of the V's we can see that

$$G(x) = \begin{cases} \frac{1}{2}F(w) + V_+(w,x) & \text{if } x \geq w, \\ \frac{1}{2}F(w) - V_+(x,w) & \text{if } x < w, \end{cases}$$
$$H(x) = \begin{cases} \frac{1}{2}F(w) + V_-(w,x) & \text{if } x \geq w, \\ \frac{1}{2}F(w) - V_-(x,w) & \text{if } x < w, \end{cases}$$

works for any $w \in I$.