MAU22200 Lecture 56

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First Fundamental Theorem of Calculus (1/5)

Here is our version of the First Fundamental Theorem of Calculus for Lebesgue integration:

Suppose $f : \mathbf{R} \to \mathbf{R}$ is integrable and

$$F(y) = \int_{x \in (-\infty, y)} f(x)\, d\mu(x).$$

Then F is continuous. For almost all y in $\mathbf{R}$ F is differentiable at y and $F'(y) = f(y)$.

I stated this last time with integration over $(-\infty, y]$. Either version is fine. It never matters for Lebesgue integrals whether you include the endpoints of an interval or not, since they have measure zero. There are other measures, e.g. Dirac measures, on $\mathbf{R}$ where you have to be more careful though.

First Fundamental Theorem of Calculus (2/5)

The following may be less familiar, but is equivalent:

Suppose $f : [a, b] \to \mathbf{R}$ is integrable. Then

$$\lim_{h \searrow 0} \frac{1}{h} \int_{x \in [y, y+h]} f(x)\, dm(x) = f(y),$$

$$\lim_{h \searrow 0} \frac{1}{h} \int_{x \in [y-h, h]} f(x)\, dm(x) = f(y),$$

and

$$\lim_{h \searrow 0} \frac{1}{2h} \int_{x \in [y-h, y+h]} f(x)\, dm(x) = f(y)$$

for almost all $x \in [a, b]$.

The relation between these is

$$\frac{F(z) - F(y)}{z - y} = \begin{cases} \frac{1}{h} \int_{x \in [y, y+h]} f(x)\, dm(x) & \text{if } z > y, \\ \frac{1}{h} \int_{x \in [y-h, y]} f(x)\, dm(x) & \text{if } z < y, \end{cases}$$

where $h = |z - y|$.

First Fundamental Theorem of Calculus (3/5)

Similarly, the usual version of the First Fundamental Theorem of Calculus for Riemann integration is equivalent to

Suppose $f : [a, b] \to \mathbf{R}$ is continuous. Then

$$\lim_{h \searrow 0} \frac{1}{h} \int_{y}^{y+h} f(x)\,dx = f(y)$$

for all $y \in [a, b)$,

$$\lim_{h \searrow 0} \frac{1}{h} \int_{y-h}^{y} f(x)\,dx = f(y)$$

for all $y \in (a, b]$, and

$$\lim_{h \searrow 0} \frac{1}{2h} \int_{y-h}^{y+h} f(x)\,dx = f(y)$$

for all $y \in (a, b)$.

Markov's Inequality (1/2)

The plan is to deduce the alternate version for Lebesgue integrals from the alternate version for Riemann integrals plus a few other pieces:

- Markov's Inequality
- The Hardy-Littlewood maximal inequality
- A density result for compactly supported continuous functions

Markov's Inequality is Proposition 10.2.1 from the notes:

> *Suppose* $(X, \mathcal{B}, \mu)$ *is a measure space,* $g \colon X \to [0, +\infty]$ *is integrable and* $\lambda > 0$. *Then*
>
> $$\mu(E_\lambda) \leq \frac{1}{\lambda} \int_{x \in X} g(x)\, d\mu(x)$$
>
> *where*
>
> $$E_\lambda = \{x \in X \colon g(x) \geq \lambda\}.$$

Roughly, the set where an integrable function is large must be small.

Markov's Inequality (2/2)

Markov's Inequality is essentially trivial, but often useful. Recall that

$$E_\lambda = \{x \in X : g(x) \geq \lambda\}.$$

Define $f = \lambda \chi_{E_\lambda}$. Then $0 \leq f(x) \leq g(x)$ for all x. So

$$\lambda \mu(E_\lambda) = \int_{x \in X} f(x)\, d\mu(x) \leq \int_{x \in X} g(x)\, d\mu(x)$$

Dividing by λ gives

$$\mu(E_\lambda) \leq \frac{1}{\lambda} \int_{x \in X} g(x)\, d\mu(x).$$

The Hardy-Littlewood Maximal Inequality

The Hardy-Littlewood Maximal Inequality, Proposition 10.2.5 in the notes, is a variant of Markov's Inequality.

Suppose $f : \mathbf{R} \to \mathbf{R}$ *is integrable. For every* $\lambda > 0$ *we have*

$$\mu(E_\lambda) \leq \frac{1}{\lambda} \int_{x \in \mathrm{R}} |f(x)| \, dm(x),$$

where

$$E_\lambda = \left\{ y \in \mathbf{R} \colon \sup_{h>0} \frac{1}{h} \int_{x \in [y, y+h]} |f(x)| \, dm(x) \geq \lambda \right\}.$$

The bound on the right hand side is the same as Hardy with $g = |f|$, but the set E_λ on the left hand side is different. This inequality is much harder to prove than Markov's. The proof isn't particularly long, but it's far from obvious. See the notes for details.

The density result (1/3)

The third piece of our puzzle is the density result, Proposition 10.2.6 in the notes:

> *Suppose $(X, \mathcal{T})$ is a locally compact Hausdorff space and μ is a Radon measure on X. If $f: X \to \mathbf{R}$ is integrable and $\epsilon > 0$ then there is a compactly supported continuous function $g: X \to \mathbf{R}$ such that*
>
> $$\int_{x \in X} |f(x) - g(x)| \, d\mu(x) < \epsilon.$$

This says that integrable functions can be approximated arbitrarily well by compactly supported continuous functions, in some appropriate sense. This sense can be described by a pseudometric coming from the seminorm

$$\|h\|_{L^1(X)} = \int_{x \in X} |h(x)| \, d\mu(x)$$

The density result (2/3)

A seminorm is a function on a vector space which satisfies all the properties of a norm, except we allow non-zero vectors to have zero norm.

From any seminorm we can construct a pseudometric, e.g.

$$d_{L^1(X)}(f, g) = \|f - g\|_{L^1(X)}.$$

A pseudometric is a function which satisfies all the properties of a metric, except we allow distinct points to have zero distance.

From a pseudometric we can construct balls and a topology, just as for a metric. This topology is not Hausdorff unless our pseudometric is a metric, but it is a topology.

The proposition on the previous slide says that the space of compactly supported continuous functions on X is a dense subset of the space of integrable functions, with the topology coming from the $L^1(X)$ seminorm, i.e. that the closure of the former space is the latter space.

The density result (3/3)

A sketch of the proof is as follows:

- Any integrable function can be approximated arbitrarily well, in the $L^1(X)$ seminorm, by a semisimple function.
- Any semisimple function can be arbitrarily well by a simple function.
- Every simple function is a linear combination of characteristic functions.
- Any characteristic function can be approximated arbitrarily well by a compactly supported continuous function.

The first three are essentially definitions. For the last one we use the fact that our measure was assumed to be a Radon measure, i.e. every Borel set has a compact subset which is not much smaller than it and an open superset which is not much larger than it. There is then a compactly supported continuous function which is equal to 1 on the compact set and equal to 0 outside the open set. This function is a good approximation, in the $L^1(X)$ sense, to the characteristic function of the Borel set.

First Fundamental Theorem of Calculus (4/5)

We put these pieces together as follows. We're given an integrable f. For any $\lambda, \epsilon > 0$ we use the density result to find a g such that

$$\int_{x \in \mathrm{R}} |f(x) - g(x)| \, dm(x) < \epsilon$$

We use the First Fundamental Theorem of Calculus (Riemann version) to get

$$\left| g(x) - \frac{1}{h} \int_{s \in [x, x+h]} g(s) \, dm(s) \right| < \lambda$$

for sufficiently small h. We use Markov's Inequality to get

$$|f(x) - g(x)| < \lambda$$

except on a set of measure at most ϵ/λ.

First Fundamental Theorem of Calculus (5/5)

We use the Hardy-Littlewood Maximal Inequality to get

$$\left| \frac{1}{h} \int_{s \in [x, x+h]} f(s)\, dm(s) - \frac{1}{h} \int_{s \in [x, x+h]} g(s)\, dm(s) \right| < \lambda$$

except on a set of measure at most ϵ/λ, for all sufficiently small h. Now we have

$$\left| f(x) - \frac{1}{h} \int_{s \in [x, x+h]} f(s)\, dm(s) \right| < 3\lambda$$

for all sufficiently small h, off of a set of measure $2\epsilon/\lambda$. We have this for every $\epsilon > 0$, so we get the inequality above off of a set of measure 0. It then follows that

$$\lim_{h \to 0^+} \frac{1}{h} \int_{s \in [x, x+h]} f(s)\, dm(s) = f(x)$$

off of this set of measure zero.