MAU22200 Lecture 55

John Stalker

Trinity College Dublin

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Extension of a measure by zero (1/2)

I want to make more explicit two constructions which I used implicitly in the last lecture: extension of a measure by zero to a superset and restriction of a measure to a subset.

Suppose $(Y, \mathcal{B}_Y)$ *is a measurable space* $X \in \mathcal{B}_Y$ *and*

$$\mathcal{B}_X = \{E \in \mathcal{B}_Y : E \subseteq X\}$$

Suppose μ_X *is a measure on* $(X, \mathcal{B}_X)$*. Define* $\mu_Y : \mathcal{B}_Y \to [0, +\infty]$ *by*

$$\mu_Y(E) = \mu_X(X \cap E).$$

Then $(Y, \mathcal{B}_Y, \mu_Y)$ *is a measure space. Also,*

$$\int_{s \in X} f(s)\, d\mu_X(s) = \int_{s \in Y} f(s)\, d\mu_Y(s)$$

for every integrable function f *on* $(Y, \mathcal{B}_Y, \mu_Y)$*.*

Extension of a measure by zero (2/2)

The proof that $(Y, \mathcal{B}_Y, \mu_Y)$ is a straightforward verification that μ_Y is a measure on $(Y, \mathcal{B}_Y)$. The statement about the integrals is a consequence of two facts:

- The inclusion function $j\colon X \to Y$ is a morphism of measure spaces from $(X, \mathcal{B}_X, \mu_X)$ to $(Y, \mathcal{B}_Y, \mu_Y)$, and
- morphisms preserve integrals, in the sense that if $j\colon X \to Y$ is a morphism then the integral of f with respect to $(Y, \mathcal{B}_Y, \mu_Y)$ is equal to the integral of $f \circ j$ with respect to $(X, \mathcal{B}_X, \mu_X)$.

The first of these is a straightforward verification. The second is a generalisation of the theorem proved earlier about refinements. See the notes for details.

Restriction of a measure (1/2)

Suppose $(Y, \mathcal{B}_Y, \mu_Y)$ *is a measure space and* $X \in \mathcal{B}_Y$*.*
Define

$$\mathcal{B}_X = \{E \in \mathcal{B}_Y : E \subseteq X\}$$

and define $\mu_X : \mathcal{B}_X \to [0, +\infty]$ *by*

$$\mu_X(E) = \mu_Y(X \cap E).$$

Then $(X, \mathcal{B}_X, \mu_X)$ *is a measure space and*

$$\int_{t \in X} g(t)\, d\mu_X(t) = \int_{t \in Y} \chi_X(t) g(t)\, d\mu_Y(t)$$

for every integrable function g *on* $(Y, \mathcal{B}_Y, \mu_Y)$*.*

Restriction of a measure (2/2)

The proofs that $\mathcal{B}_X$ is a σ-algebra and μ_X is a measure are straightforward verifications. The proof that this integrals are equal is done by proving it first for semisimple functions and then using the fact that integrable functions are those which can be approximated arbitrarily well by semisimple functions. See the notes for details.

Extending a measure by zero to a superset and then restricting back to the original subset gives us back the measure we started with.

Restricting to a subset and then extending by zero back to the original set does not. If we start with a measure μ_Y on Y and restrict it to a measure μ_X on $X \subseteq Y$ and then extend this by zero to Y we get a measure ν on Y, where $\nu(E) = \mu_Y(X \cap E)$.

We'll always use the restriction of Lebesgue measure as our measure on subsets of $\mathbf{R}$ unless some other measure is specified.

The Fundamental Theorem of Calculus (1/8)

The Fundamental Theorem of Calculus is really two theorems.

The first one is

> *Suppose $f : [a, b] \to \mathbf{R}$ is continuous. Define $F : [a, b] \to \mathbf{R}$ by*
>
> $$F(y) = \int_a^y f(x)\,dx.$$
>
> *Then F is differentiable and $F'(x) = f(x)$ for all $x \in [a, b]$.*

The second one is

> *Suppose $F : [a, b] \to \mathbf{R}$ is differentiable and F' is Riemann integrable. Then*
>
> $$\int_a^b F'(x)\,dx = F(b) - F(a).$$

These are of course the versions for Riemann integration. We'd like versions for Lebesgue integration.

The Fundamental Theorem of Calculus (2/8)

For the first theorem we want to replace the hypothesis that f is continuous with the hypothesis that f is integrable. Of course we also replace the Riemann integral with

$$F(y) = \int_a^y f(x)\,dx.$$

with the Lebesgue integral

$$F(y) = \int_{x\in[a,y]} f(x)\,dm(x).$$

For technical reasons it's more common to consider functions on $\mathbf{R}$ rather than $[a, b]$ and to use

$$F(y) = \int_{x\in(-\infty,y]} f(x)\,dm(x).$$

The Fundamental Theorem of Calculus (3/8)

What about the conclusion of the theorem? Can we expect that F is differentiable and everywhere and $F'(x) = f(x)$ for all $x \in \mathbf{R}$? That doesn't work. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x > 1. \end{cases}$$

This is a simple function so it's easy to compute the integral:

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

$F'(x) = f(x)$ wherever F is differentiable, but F isn't differentiable at the points 0 and 1. $m(\{0, 1\}) = 0$ so F is differentiable almost everywhere. The example suggests the correct conclusion should be "F is continuous and for almost all $x \in \mathbf{R}$ the function F is differentiable at x and $F'(x) = f(x)$." It is.

The Fundamental Theorem of Calculus (4/8)

For the Second Fundamental Theorem of Calculus we want to replace the hypothesis that F' is Riemann integrable with the hypothesis that it's Lebesgue integrable. If we want to be able to apply it to functions like the one from the previous slide we should also only assume F is differentiable almost everywhere.
What about the conclusion of the theorem? We should replace the Riemann integral

$$\int_a^b F'(x)\,dx$$

with the Lebesgue integral

$$\int_{x\in[a,b]} F'(x)\,dm(x).$$

The integrand is only defined almost everywhere, but that's okay.

The Fundamental Theorem of Calculus (5/8)

Is it true that

$$\int_{x \in [a,b]} F'(x)\, dm(x) = F(b) - F(a)?$$

No, or at least not without additional hypotheses. Consider

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

F is differentiable except at 0, so almost everywhere. In fact $F'(x) = 0$ for almost all x.

$$\int_{x \in [a,b]} F'(x)\, dm(x) = 0$$

no matter what a and b are. But $F(b) - F(a) = 1$ if $a < 0$ and $b \geq 0$.

The Fundamental Theorem of Calculus (6/8)

We shouldn't be surprised by this failure. The First Fundamental Theorem of Calculus tells us that

$$\int_{x\in[a,b]} F'(x)\,dm(x)$$

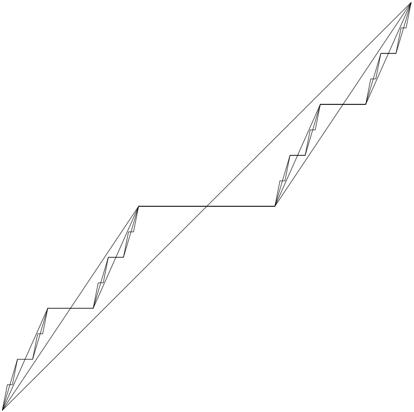
depends continuously on the endpoints a and b. $F(b) - F(a)$ doesn't depend continuously on the endpoints unless F is continuous, so continuity of F must be a necessary condition for

$$\int_{x\in[a,b]} F'(x)\,dm(x) = F(b) - F(a)$$

to hold. Is it a sufficient condition? Unfortunately not.

The Fundamental Theorem of Calculus (7/8)

There's a function obtained as a limit of piecewise linear functions as in this picture:



It's sometimes called the *Cantor function* or the *devil's staircase*. It's continuous and is differentiable with derivative zero except on the Cantor set, so almost everywhere. But it's not constant.

The Fundamental Theorem of Calculus (8/8)

We're looking for a hypothesis intermediate between "continuous everywhere and differentiable almost everywhere" and "differentiable everywhere". "uniformly continuous" turns out not to be enough. The Cantor function is uniformly continuous. "Lipschitz continuous" *is* enough. It not only excludes the Cantor function, it is enough to get the desired equation

$$\int_{x \in [a,b]} F'(x)\, dm(x) = F(b) - F(a)$$

There is an even weaker hypothesis which would work, called absolute continuity, but it's not worth the effort.