MAU22200 Lecture 54

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15 March 2022

A topological lemma (1/2)

> *Suppose* $(X, \mathcal{T})$ *is a locally compact Hausdorff topological space and* $K \in \wp(X)$ *is compact,* $U \in \wp(X)$ *is open and* $K \subseteq U$*. Then there is a continuous compactly supported function* $g \colon X \to [0, 1]$ *such that* $g(x) = 1$ *for all* $x \in K$ *and* $g(x) = 0$ *for all* $x \in X \setminus U$*.*

This is a variant of Urysohn's Lemma:

> *Suppose* $(X, \mathcal{T})$ *is a normal topological space and* A *and* B *are closed subsets of* X *such that* $A \cap B = \varnothing$*. Then there is a continuous function* $f \colon X \to [0, 1]$ *such that* $f(x) = 0$ *for all* $x \in A$ *and* $f(x) = 1$ *for all* $x \in B$*.*

Compact Hausdorff spaces are both normal and locally compact.
If X is compact then the two are equivalent: just take $B = K$ and $A = X \setminus U$.
To get the locally compact case, take a compact neighbourhood W_x of x for each $x \in X$.

A topological lemma (2/2)

W_x is compact neighbourhood of x means W_x is compact and there is an open V_x such that $x \in V_x$ and $V_x \subseteq W_x$. The V_x with $x \in K$ are an open cover of K so there is a finite subcover $V_{x_1}, \ldots, V_{x_m}$. Then

$$K \subseteq U \cap V \subseteq V \subseteq W,$$

where $V = \bigcup_{i=1}^{m} V_{x_i}$ and $W = \bigcup_{i=1}^{m} W_{x_i}$. W is compact and $L = W \setminus (U \cap V)$ is a closed subset of W. Applying Urysohn's Lemma gives a continuous $f \colon W \to [0, 1]$ such that $f(x) = 1$ if $x \in K$ and $f(x) = 0$ if $x \in L$. If $x \notin U$ then $x \in L$ so $f(x) = 0$. We then extend f to a function g on X by $g(x) = f(x)$ if $x \in W$ and $g(x) = 0$ if $x \notin W$. This g is continuous.

Another topological lemma

Suppose $(X, \mathcal{T})$ is a locally compact σ-compact Hausdorff topological space. Then there is a sequence K_0, K_1, ... of compact subsets such that $K_m \subseteq K_{m+1}^\circ$ for all m and $\bigcup_{m=0}^{\infty} K_m = X$.

X is σ-compact. In other words, there are compact A_0, A_1, ..., such that $X = \bigcup_{m=0}^{\infty} A_m$. X locally compact, so we have V_x, W_x as before. Define K_m inductively, starting with $K_0 = \varnothing$. $A_m \cup K_m$ is compact. The V_x with $x \in A_m \cup K_m$ are an open cover of it so there are $x_{m,0}$, ..., x_{m,n_m} such that

$$A_m \cup K_m \subseteq V_m = \bigcup_{j=0}^{n_m} V_{m,j}.$$

Set $K_{m+1} = \bigcup_{j=0}^{n_m} W_{m,j}$. Then $V_m \subseteq K_{m+1}$ and V_m is open so $V_m \subseteq K_{m+1}^\circ$. Also $A_m \subseteq K_{m+1}$ and $\bigcup_{m=0}^{\infty} A_m = X$ so $\bigcup_{m=0}^{\infty} K_m = X$.

A measure theory lemma (1/3)

The following is like a Monotone Convergence Theorem for measures:

Suppose $\mathcal{B}$ is a σ algebra on a set X and $\mu_0, \mu_1, \ldots$ are measures on $(X, \mathcal{B})$ which are monotone in the sense that for all $E \in \mathcal{B}$ then

$$\mu_j(E) \leq \mu_k(E)$$

whenever $j \leq k$. Let

$$\mu(E) = \lim_{j \to \infty} \mu_j(E).$$

Then μ is a measure on $(X, \mathcal{B})$ and

$$\int_{x \in X} f(x)\, d\mu(x) = \lim_{j \to \infty} \int_{x \in X} f(x)\, d\mu_j(x).$$

A measure theory lemma (2/3)

It's easy to see that μ is a measure. First $\mu(\varnothing) = 0$. Suppose E_0, E_1, ..., are disjoint elements of $\mathcal{B}$. Then

$$\mu\left(\bigcup_{k=0}^{\infty} E_k\right) = \lim_{j\to\infty} \mu_j\left(\bigcup_{k=0}^{\infty} E_k\right) = \lim_{j\to\infty} \sum_{k=0}^{\infty} \mu_j\left(E_k\right)$$
$$= \sum_{k=0}^{\infty} \lim_{j\to\infty} \mu_j\left(E_k\right) = \sum_{k=0}^{\infty} \mu\left(E_k\right).$$

It's also easy to check

$$\int_{x\in X} f(x)\,d\mu(x) = \lim_{j\to\infty} \int_{x\in X} f(x)\,d\mu_j(x).$$

when f is semisimple. There are $\mathcal{Q}$, φ such $f(x) = \varphi(E)$ if $x \in E \in \mathcal{Q}$.

A measure theory lemma (3/3)

$$\int_{x\in X} f(x)\, d\mu(x) = \sum_{E\in\mathcal{Q}} \varphi(E)\mu(E) = \sum_{E\in\mathcal{Q}} \varphi(E) \lim_{j\to\infty} \mu_j(E)$$
$$= \lim_{j\to\infty} \sum_{E\in\mathcal{Q}} \varphi(E)\mu_j(E) = \lim_{j\to\infty} \int_{x\in X} f(x)\, d\mu_j(x).$$

The integral of any function is determined by integrals of semisimple functions so we get

$$\int_{x\in X} f(x)\, d\mu(x) = \lim_{j\to\infty} \int_{x\in X} f(x)\, d\mu_j(x).$$

for all f.

Uniqueness (1/2)

Suppose $(X, \mathcal{T})$ is a locally compact Hausdorff space. Suppose I is a linear transformation from the vector space of continuous compactly supported functions from X to $\mathbf{R}$ such that $I(g) \geq 0$ whenever g is such that $g(x) \geq 0$ for all $x \in X$. Then there is at most one Radon measure μ on X such that

$$I(g) = \int_{x \in X} g(x)\, d\mu(x)$$

for all continuous compactly supported g.

Suppose μ_1 and μ_2 are Radon measures such that

$$\int_{x \in X} g(x)\, d\mu_1(x) = I(g) = \int_{x \in X} g(x)\, d\mu_2(x)$$

for all continuous compactly supported g. Suppose K is compact, U is open and $K \subseteq U$.

Uniqueness (2/2)

By our first topological lemma there is a continuous compactly supported function g such that such that $g(x) = 1$ for $x \in K$ and $g(x) = 0$ for $x \notin U$. In other words, $\chi_K(x) \leq g(x) \leq \chi_U(x)$ for all $x \in U$. Therefore

$$\begin{aligned}\mu_1(K) &= \int_{x\in X} \chi_K(x)\, d\mu_1(x) \leq \int_{x\in X} g(x)\, d\mu_1(x) = I(g) \\ &= \int_{x\in X} g(x)\, d\mu_2(x) \leq \int_{x\in X} \chi_U(x)\, d\mu_2(x) = \mu_2(U).\end{aligned}$$

Thus $\mu_1(K) \leq \mu_2(U)$ whenever K is compact, U is open and $K \subseteq U$. μ_1 is a Radon measure so $\mu_1(U) = \sup \mu_1(K)$, where the supremum is over all compact subsets K of U so $\mu_1(U) \leq \mu_2(U)$. The same argument works with μ_1 and μ_2 swapped, so $\mu_2(U) \leq \mu_1(U)$ and hence $\mu_1(U) = \mu_2(U)$. μ_1 and μ_2 are both Radon measures so $\mu_1(E) = \inf \mu_1(U)$ and $\mu_2(E) = \inf \mu_2(U)$ for any Borel set E, where the infimum in both cases is over open supersets U of E. The right hand sides are equal, so the left hand sides are equal: $\mu_1(E) = \mu_2(E)$.

The Riesz Representation Theorem (1/3)

Suppose $(X, \mathcal{T})$ is a locally compact, σ-compact Hausdorff topological space. Suppose I is a linear transformation from the vector space of continuous compactly supported functions from X to $\mathbf{R}$ such that $I(g) \geq 0$ whenever $g(x) \geq 0$ for all $x \in X$. Then there is a unique Radon measure μ on X such that all continuous compactly supported functions g are integrable and

$$\int_{x \in X} g(x)\, d\mu(x) = I(g).$$

Note that we already have uniqueness in this level of generality, we just need existence.

The Riesz Representation Theorem (2/3)

Let K_0, K_1, ... be as in the second topological lemma. By our first topological lemma there are continuous $h_n \colon X \to [0,1]$ such that $h_n(x) = 1$ if $x \in K_n$ and $h_n(x) = 0$ if $x \notin K_{n+1}^\circ$. Define $I_n(g)$ for compactly supported continuous g from K_{n+1} to $\mathbf{R}$ by $I_n(g) = I(gh_n)$. By the weak version of the Riesz Representation Theorem there's a measure μ_n on K_{n+1} such that $\int_{x \in K_{n+1}} g(x)\, d\mu_n(x) = I_n(g)$. We can extend this measure to be zero outside K_{n+1}, so $\int_{x \in X} g(x)\, d\mu_n(x) = I_n(g)$ for all compactly supported continuous g from X to $\mathbf{R}$. Our measure theory lemma gives a measure μ such that

$$\int_{x \in X} f(x)\, d\mu(x) = \lim_{n \to \infty} \int_{x \in X} f(x)\, d\mu_n(x).$$

The Riesz Representation Theorem (3/3)

If g is compactly supported then it's supported in some K_{n+1} so $g = gh_k$ for all $k > n$. Then $I(g) = I(gh_k) = I_k(g)$ and

$$\begin{aligned} I(g) &= \lim_{k\to\infty} I_k(g) = \lim_{k\to\infty} \int_{x\in X} g(x)\, d\mu_k(x) \\ &= \int_{x\in X} g(x)\, d\mu(x). \end{aligned}$$