MAU22200 Lecture 53

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Riesz Representation Theorem for compact X (1/6)

Suppose $(X, \mathcal{T})$ is a compact Hausdorff topological space. Suppose I is a linear transformation from the vector space of continuous compactly supported functions from X to $\mathbf{R}$ such that $I(g) \geq 0$ whenever $g(x) \geq 0$ for all $x \in X$. Then there is a Radon measure μ on X such that all continuous compactly supported functions g are integrable and

$$\int_{x \in X} g(x)\, d\mu(x) = I(g).$$

This is the Riesz Representation Theorem stated, but not proved, last time, with two simplifications. We're now assuming X is compact rather than locally compact and σ-compact, so we've strengthened the hypotheses. We're only claiming that μ exists, not that it's unique, so we've weakened the conclusion. So this is a weaker theorem than the full Riesz Representation Theorem. It contains the hardest parts though. We'll fill in the rest once we have this version.

Riesz Representation Theorem for compact X (2/6)

The proof is quite long, and far from straightforward. You don't really need to pay too much attention to it, but if you want to read it then you might find this outline helpful.

- We prove a monotonicity property for I: If $f(x) \leq g(x)$ for all $x \in X$ then $I(f) \leq I(g)$.
- We define a function J from the set of bounded non-negative lower semicontinuous functions by $J(g) = \sup I(f)$, where the supremum is over all continuous functions f such that $0 \leq f(x) \leq g(x)$ for all $x \in X$.
- We show that $0 \leq J(g) < +\infty$ for all such g.
- We show that when $I(g)$ and $J(g)$ are defined, i.e. when g is continuous and non-negative, $I(g) = J(g)$.

Riesz Representation Theorem for compact X (3/6)

- We prove a monotonicity property for J: If $g(x) \leq h(x)$ for all $x \in X$ then $J(g) \leq J(h)$.
- We prove that J is homogeneous in the sense that if $c \geq 0$ and g is bounded, non-negative and lower semicontinuous then $J(cg) = cJ(g)$.
- We prove that J is finitely superadditive in the sense that if $g_0, \ldots, g_m$ are bounded, non-negative and lower semicontinuous then

$$J\left(\sum_{k=0}^{m} g_k\right) \geq \sum_{k=0}^{m} J(g_k).$$

Riesz Representation Theorem for compact X (4/6)

- We prove that J is countably subadditive in the sense that if $g_0, g_1, \ldots,$ are bounded, non-negative and lower semicontinuous then

$$J\left(\sum_{k=0}^{\infty} g_k\right) \leq \sum_{k=0}^{\infty} J(g_k).$$

This is in fact a special case of a slightly more general statement. If $g_0, g_1, \ldots,$ are bounded, non-negative and lower semicontinuous and

$$f(x) \leq \sum_{k=0}^{\infty} g_k(x)$$

for all $x \in X$, where f is also bounded, non-negative and lower semicontinuous, then

$$J(f) \leq \sum_{k=0}^{\infty} J(g_k).$$

Riesz Representation Theorem for compact X (5/6)

- We define a function ν on the set of open subsets of X by
$$\nu(E) = J(\chi_E)$$
and a function ν on the set of closed subsets of X by
$$\nu(E) = J(\chi_X) - J(\chi_{X \setminus E})$$
and show that if E is both open and closed then the two definitions agree.
- We prove a monotonicity property of ν: If $V \subseteq U$ and $\nu(V)$ and $\nu(U)$ are both defined, i.e. if each of U and V is open or closed, then $\nu(V) \leq \nu(U)$.
- We define functions μ^- and μ^+ on $\wp(X)$ by
$$\mu^-(E) = \sup \nu(V)$$
and
$$\mu^+(E) = \inf \nu(U),$$
where the supremum is over closed V such that $V \subseteq E$ and the infimum is over open U such that $E \subseteq U$.

Riesz Representation Theorem for compact X (6/6)

- We define $\mathcal{B}$ to be the set of $E \in \wp(X)$ such that $\mu^+(E) \leq \mu^-(E)$, which then implies $\mu^+(E) = \mu^-(E)$.
- We define a function μ on $\mathcal{B}$ by $\mu(E) = \mu^+(E) = \mu^-(E)$.
- We show that $\mathcal{B}$ is a σ-algebra on X and μ is a measure on $(X, \mathcal{B})$.
- We show that if E is closed then $E \in \mathcal{B}$ and $\mu(E) = \nu(E)$.
- We show that if E is open then $E \in \mathcal{B}$ and $\mu(E) = \nu(E)$.
- We show that $\mathcal{B}$ is a superset of the Borel σ-algebra.
- We show that if f is bounded, positive and lower semicontinuous then
$$\int_{x \in X} f(x)\, d\mu(x) = J(f).$$
- We show that if f is continuous then
$$\int_{x \in X} f(x)\, d\mu(x) = I(f).$$

Some topological lemmas

To get the uniqueness of the measure μ and to extend the theorem from compact spaces to locally compact σ-compact spaces we need some topological lemmas.

> *Suppose $(X, \mathcal{T})$ is a locally compact Hausdorff topological space and $K \in \wp(X)$ is compact, $U \in \wp(X)$ is open and $K \subseteq U$. Then there is a continuous compactly supported function $g \colon X \to [0, 1]$ such that $f(x) = 1$ for all $x \in K$ and $f(x) = 0$ for all $x \in X \setminus U$.*

and

> *Suppose $(X, \mathcal{T})$ is a locally compact σ-compact Hausdorff topological space. Then there is a sequence $K_0, K_1, \ldots$ of compact subsets such that $K_m \subseteq K_{m+1}^\circ$ for all m and $\bigcup_{m=0}^\infty K_m = X$.*

I'll prove these next time and use them to get the full version of the Riesz Representation Theorem.