MAU22200 Lecture 52

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Lebesgue integration

We have a whole theory of integration on measure spaces, but so far our only examples of measure spaces are rather trivial, e.g. counting measure or Dirac measure.

The most commonly used measure space is $(\mathbf{R}^n, \mathcal{B}, m)$, where $\mathcal{B}$ is the Lebesgue σ-algebra on $\mathbf{R}^n$ and m is the *Lebesgue measure*.

Let's just consider $n = 1$ for the moment.

$\mathcal{B}$ is a superset of the Borel σ-algebra, which in turn is a superset of the Jordan (Boolean) algebra, the completion of $\mathcal{I}$. The Lebesgue measure m is an extension of the Jordan content, i.e. if $E \in \mathcal{J}$ then the Lebesgue measure of E, $m(E)$, is equal to its Jordan content $\mu_{\mathcal{J}}(E)$. In other words, $(\mathbf{R}, \mathcal{B}, m)$ is a refinement of $(\mathbf{R}, \mathcal{J}, \mu_{\mathcal{J}})$.

The Lebesgue integral is also an extension of the Riemann integral, in the sense that if g is Riemann integrable then it's Lebesgue integrable, i.e. integrable with respect to $(\mathbf{R}, \mathcal{B}, m)$ and $\int_{x \in \mathrm{R}} g(x)\, dm(x)$ is equal to the Riemann integral of g. In particular, compactly supported continuous functions are Lebesgue integrable.

Introducing the Riesz Representation Theorem

There are several ways to obtain Lebesgue measure, all of which are messy. We'll do it in a way which automatically gives the Lebesgue integral as an extension of the Riemann integral. This uses the *Riesz Representation Theorem*:

> *Suppose* $(X, \mathcal{T})$ *is a locally compact,* σ*-compact Hausdorff topological space. Suppose* I *is a linear transformation from the vector space of continuous compactly supported functions from* X *to* $\mathbf{R}$ *such that* $I(g) \geq 0$ *whenever* $g(x) \geq 0$ *for all* $x \in X$*. Then there is a unique Radon measure* μ *on* X *such that all continuous compactly supported functions* g *are integrable and*
>
> $$\int_{x \in X} g(x)\, d\mu(x) = I(g).$$

The function I taking each continuous compactly supported function to its Riemann integral is a linear transformation such that $I(g) \geq 0$ whenever $g \geq 0$, so the theorem applies.

Other applications of the theorem (1/2)

We're mostly interested in Riemann integral case, where the theorem gives us the existence of Lebesgue measure, but there are other important applications:

- There's an analogue of Riemann integration in $\mathbf{R}^n$ for $n > 1$. We'll see this in a couple of weeks. The Riesz Representation Theorem means that once we have an analogue of Riemann integration in $\mathbf{R}^n$ we get an analogue of Lebesgue integration in $\mathbf{R}^n$ without further work.
- There are other types of integral of interest in $\mathbf{R}^n$, e.g. surface integrals. The RRT allows us to get all the useful properties of the Lebesgue integral, e.g. the Monotone Convergence Theorem, Fatou's Lemma or the Dominated Convergence Theorem, for these integrals, if we know how to define them on compactly supported continuous functions.

Other applications of the theorem (2/2)

- Suppose V is a real-valued random variable. Its *cumulative probability distribution* is the function

$$F_V(x) = P(V \leq x).$$

We can define the *Riemann-Stieltjes integral* of functions on **R** by mimicking the construction of the Riemann integral, but using the difference of the F_V values at the endpoints of the interval in place of its length. The Riesz Representation Theorem gives us a corresponding measure μ_V on **R** and a corresponding integral, called the *Lebesgue-Stieltjes integral*. The probabilistic interpretation of this integral is that

$$\int_{x \in \mathrm{R}} g(x)\, d\mu_V(x)$$

is the *expected value* of $g(V)$.

Comments on the proof

The proof is long. To break it up a bit I've separated out the case of compact X, and separated the existence of μ from its uniqueness. The hard part is existence in the compact case.

Most of the proof not worth knowing in detail. In principle it gives you a construction of Lebesgue measure, but you never want to use this construction, only the fact that Lebesgue measure is a Radon measure which extends Jordan content and that the corresponding integral extends the Riemann integral.

I'll give a sketch of the proof next time, but even the sketch is not worth committing to memory. There are a few concepts which are used which are worth remembering, especially semicontinuity. I've put the semicontinuity results in their own section, before the proof.

Semicontinuity (1/5)

Suppose $(X, \mathcal{T})$ is a topological space and $f: X \to \mathbf{R}$ is a function. f is said to be *lower semicontinuous* if $f^*((a, +\infty)) \in \mathcal{T}$ for all $a \in \mathbf{R}$. f is said to be *upper semicontinuous* if $f^*((-\infty, b)) \in \mathcal{T}$ for all $b \in \mathbf{R}$.
There are various ways to think about semicontinuous functions. For example, there's a topology $\mathcal{T}_+$ on $\mathbf{R}$ consisting of the sets of the form $(a, +\infty)$, $\varnothing$ and $\mathbf{R}$. It's not a very nice topology. It's not Hausdorff, for example, but it is a topology. It's a weaker topology than the usual one. f is lower semicontinuous if and only if it is continuous as a function from $(X, \mathcal{T})$ to $(\mathbf{R}, \mathcal{T}_+)$. This point of view has some uses, e.g. if f is continuous with respect to the usual (metric) topology then it is semicontinuous. Or if $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, $g: X \to Y$ is continuous and $f: Y \to \mathbf{R}$ is lower semicontinuous then $f \circ g$ is lower semicontinuous.

Semicontinuity (2/5)

There's another topology $\mathcal{T}_-$ on $\mathbf{R}$ consisting of the sets of the form $(-\infty, b)$, $\varnothing$ and $\mathbf{R}$. f is upper semicontinuous if and only if it is continuous as a function from $(X, \mathcal{T})$ to $(\mathbf{R}, \mathcal{T}_-)$.

$\mathcal{T}_+ \cup \mathcal{T}_-$ is not a topology on $\mathbf{R}$, but it does generate a topology, because any set of subsets generates a topology. In fact the topology that $\mathcal{T}_+ \cup \mathcal{T}_-$ generates is the usual (metric) topology on $\mathbf{R}$. It contains $(a, +\infty) \cap (-\infty, b) = (a, b)$ for any $a, b \in \mathbf{R}$, and therefore contains every union of sets of the form (a, b), i.e. every open set in the usual topology.

It follows that f is continuous, with respect to the usual topology, if and only if is both upper and lower semicontinuous. You can also prove this directly from the definition, which is what I've done in the notes. Sometimes it's convenient to prove a function is continuous by first proving it's lower semicontinuous and then proving it's upper semicontinuous. That's like proving an equation by proving two inequalities.

Semicontinuity (3/5)

I haven't given you and examples yet, but the following proposition gives plenty of them:

Suppose $(X, \mathcal{T})$ is a topological space, $E \in \wp(X)$ and χ_E is the characteristic function of E, i.e.

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Then χ_E is lower semicontinuous if and only if E is open. χ_E is upper semicontinuous if and only if E is closed.

The "only if" is easy:

$x \in E \Leftrightarrow \chi_E(x) = 1 \Leftrightarrow \chi_E(x) \in (1/2, +\infty) \Leftrightarrow x \in \chi_E^*((1/2, +\infty))$.

$\chi_E^*((1/2, +\infty))$ is open if χ_E is lower semicontinuous. Similarly,

$x \in X \backslash E \Leftrightarrow \chi_E(x) = 0 \Leftrightarrow \chi_E(x) \in (-\infty, 1/2) \Leftrightarrow x \in \chi_E^*((-\infty, 1/2))$.

$\chi_E^*(-\infty, 1/2))$ is open if χ_E is upper semicontinuous.

Semicontinuity (4/5)

The “if” isn’t much harder.

$$\chi_E^*((a, +\infty)) = \begin{cases} \varnothing & \text{if } a \geq 1, \\ E & \text{if } 0 \leq a < 1, \\ X & \text{if } a < 0. \end{cases}$$

In each case $\chi_E^*((a, +\infty))$ is open if E is open.

$$\chi_E^*((-\infty, b)) = \begin{cases} \varnothing & \text{if } b \leq 0, \\ X \setminus E & \text{if } 0 < b \leq 1, \\ X & \text{if } b > 1. \end{cases}$$

In each case $\chi_E^*((-\infty, b))$ is open if E is closed.

Semicontinuity (5/5)

Linear combinations of continuous functions are continuous. Linear combinations of upper/lower semicontinuous functions needn't be upper/lower semicontinuous. For example $-f$ is lower semicontinuous if and only if f is upper semicontinuous, and vice versa. Linear combinations of upper/lower semicontinuous functions with non-negative coefficients are upper/lower semicontinuous though.

Suppose $g = \sum_{i=1}^{m} c_i f_i$. Then

$$g^*((a, +\infty)) = \bigcup \bigcap_{i=1}^{m} f_i^*((\alpha_i, +\infty))$$

$$g^*((-\infty, b)) = \bigcup \bigcap_{i=1}^{m} f_i^*((-\infty, \beta_i)),$$

where the first union is over α's such that $\sum_{i=1}^{m} c_i \alpha_i = a$ and the second union is over β's such that $\sum_{i=1}^{m} c_i \beta_i = b$.