MAU22200 Lecture 51

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Convergence Theorems

We had three main theorems about the interchange of limits and sums:

1. The Monotone Convergence Theorem
2. Fatou's Lemma
3. The Dominated Convergence Theorem

Each of these applied to limits of nets, not just of sequences. There are three main theorems about the interchange of limits and integrals over measure spaces:

1. The Monotone Convergence Theorem
2. Fatou's Lemma
3. The Dominated Convergence Theorem

These only work over measure spaces, not content spaces. That's the main reason for introducing measure spaces. These theorem work for sequences, but not generally for nets (or filters). The MCT for integrals has a long proof, using the corresponding theorem for sums and measures. The other two theorems have shorter proofs, which are almost identical to those for sums.

Monotone Convergence Theorem (1/6)

Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f\colon \mathbf{N} \times X \to [0, +\infty]$ is a function such that

- *$f_n(x)$ is an integrable function of x for each $n \in \mathbf{N}$, and*
- *$f_n(x)$ is a monotone increasing sequence in n for each $x \in X$.*

Then

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) = \int_{x\in X} \lim_{m\to\infty} f_m(x)\, d\mu(x).$$

The second condition above implies that $\lim_{m\to\infty} f_m(x)$ exists, so the integrand on the right is well defined. It's integrable because for functions with values in $[0, +\infty]$ integrable and measurable are the same, and limits of measurable functions are measurable. The proof of the Monotone Convergence Theorem for integrals is based on the Monotone Convergence Theorem for sums and the Monotone Convergence Theorem for measures.

Monotone Convergence Theorem (2/6)

$$f_m(x) \leq \sup_{n \in \mathrm{N}} f_n(x) = \lim_{n \to \infty} f_n(x)$$

for all $m \in \mathbf{N}$. The equation between the supremum and the limit follows from the monotonicity assumption on f. Therefore

$$\int_{x \in X} f_m(x)\, d\mu(x) \leq \int_{x \in X} \lim_{n \to \infty} f_n(x)\, d\mu(x).$$

Taking the supremum over all $m \in \mathbf{N}$ we get

$$\sup_{m \in \mathrm{N}} \int_{x \in X} f_m(x)\, d\mu(x) \leq \int_{x \in X} \lim_{n \to \infty} f_n(x)\, d\mu(x).$$

If $m \leq n$ then $f_m(x) \leq f_n(x)$ for all $x \in X$ by the monotonicity assumption so

$$\int_{x \in X} f_m(x)\, d\mu(x) \leq \int_{x \in X} f_n(x)\, d\mu(x)$$

Monotone Convergence Theorem (3/6)

$$\int_{x\in X} f_m(x)\,d\mu(x) \leq \int_{x\in X} f_n(x)\,d\mu(x)$$

so $\int_{x\in X} f_m(x)\,d\mu(x)$ is a monotone sequence and therefore

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\,d\mu(x) = \sup_{m\in\mathrm{N}} \int_{x\in X} f_m(x)\,d\mu(x).$$

Thus

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\,d\mu(x) \leq \int_{x\in X} \lim_{n\to\infty} f_n(x)\,d\mu(x).$$

The name of the variable in the limit is irrelevant, so

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\,d\mu(x) \leq \int_{x\in X} \lim_{m\to\infty} f_m(x)\,d\mu(x).$$

Now we have half of what we want.

Monotone Convergence Theorem (4/6)

Suppose g is a semisimple function such that $g(x) \leq \lim_{m\to\infty} f_m(x)$ for all $x \in X$. In other words, there is a countable partition $\mathcal{Q} \subseteq \mathcal{B}$ of X such that $\wp([0,+\infty]) \subseteq g^{**}(\mathcal{B})$. Then, as we've seen, there is a $\varphi\colon \mathcal{Q} \to [0,+\infty]$ such that $g(x) = \varphi(E)$ when $x \in E$. Suppose $\kappa \in (0,1)$. Then

$$\lim_{m\to\infty} f_m(x) = \sup_{m\to\infty} f_m(x) \geq g(x) = \varphi(E) > \kappa\varphi(E)$$

for all $x \in E$. The last inequality requires $\varphi(E) \neq 0$, which we'll assume from now until further notice. Define

$$F_{m,E} = \{x \in E\colon f_m(x) > \kappa\varphi(E)\}.$$

Then $F_{m,E} \subseteq F_{n,E}$ whenever $m \leq n$ and $\bigcup_{m\in \mathrm{N}} F_{m,E} = E$. It follows from the Monotone Convergence Theorem for measures, Theorem 7.6.7, that

$$\lim_{m\to\infty} \mu(F_{m,E}) = \mu(E).$$

Monotone Convergence Theorem (5/6)

Let

$$h_m(x) = \begin{cases} \kappa\varphi(E) & \text{if } x \in F_{m,E}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_m(x) \geq h_m(x)$ for all $x \in X$ so

$$\int_{x\in X} f_m(x)\, d\mu(x) \geq \int_{x\in X} h_m(x)\, d\mu(x) = \sum_{E\in\mathcal{Q}} \kappa\varphi(E)\mu\left(F_{m,E}\right).$$

The sum is over all E and we've been assuming $\varphi(E) \neq 0$, but that's okay since the E's for which $\varphi(E) = 0$ don't contribute to any of the sums or integrals. Now

$$\begin{aligned} \lim_{m\to\infty} \sum_{E\in\mathcal{Q}} \kappa\varphi(E)\mu(F_{m,E}) &= \sum_{E\in\mathcal{Q}} \lim_{m\to\infty} \kappa\varphi(E)\mu(F_{m,E}) \\ &= \sum_{E\in\mathcal{Q}} \kappa\varphi(E)\mu(E) = \kappa \sum_{E\in\mathcal{Q}} \varphi(E)\mu(E) \\ &= \kappa \int_{x\in X} g(x)\, d\mu(x). \end{aligned}$$

Monotone Convergence Theorem (6/6)

The interchange of the sum and limit one the previous slide is justified by the Monotone Convergence Theorem for sums, Theorem 6.3.1. It follows that

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) \geq \kappa \int_{x\in X} g(x)\, d\mu(x)$$

for all $\kappa \in (0,1)$ and hence, taking the limit as κ tends to 1 from below,

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) \geq \int_{x\in X} g(x)\, d\mu(x).$$

This is true for all simple g such that $g(x) \leq \lim_{m\to\infty} f_m(x)$ so the limit on the left is greater than or equal to the supremum over all such g, i.e.

$$\begin{aligned} \lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) &\geq \underline{\int}_{x\in X} \lim_{m\to\infty} f_m(x)\, d\mu(x). \\ &= \int_{x\in X} \lim_{m\to\infty} f_m(x)\, d\mu(x). \end{aligned}$$

We already have the reverse inequality, so both sides are equal.

Fatou's Lemma (1/3)

Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f\colon \mathbf{N} \times X \to [0, +\infty]$ is a function such that $f_m(x)$ is integrable as a function of x for each $m \in \mathbf{N}$. Then

$$\int_{x \in X} \sup_{m \in \mathrm{N}} \inf_{n \geq m} f_n(s)\, d\mu(x) \leq \sup_{m \in \mathrm{N}} \inf_{n \geq m} \int_{x \in X} f_n(s)\, d\mu(x).$$

Define $g\colon \mathbf{N} \times X \to [0, +\infty]$ by $g_m(x) = \inf_{n \geq m} f_n(x)$. If $m \leq n$ then

$$\{p \in \mathbf{N} \colon p \geq n\} \subseteq \{p \in \mathbf{N} \colon p \geq m\}$$

and so $\inf_{p \geq m} f_p(x) \leq \inf_{p \geq n} f_p(x)$. In other words, if $m \leq n$ then $g_m(x) \leq g_n(x)$. It follows from the Monotone Convergence Theorem that

$$\int_{x \in X} \lim_{m \to \infty} g_m(x)\, d\mu(x) = \lim_{m \to \infty} \int_{x \in X} g_m(x)\, d\mu(x).$$

Fatou's Lemma (2/3)

$$\int_{x\in X} \lim_{m\to\infty} g_m(x)\, d\mu(x) = \lim_{m\to\infty} \int_{x\in X} g_m(x)\, d\mu(x).$$

These are monotone sequences so the limit is the same as the supremum and therefore

$$\int_{x\in X} \sup_{m\in\mathrm{N}} g_m(x)\, d\mu(x) = \sup_{m\in\mathrm{N}} \int_{x\in X} g_m(x)\, d\mu(x).$$

Now if $m \leq n$ then $g_m(x) = \inf_{p\geq m} f_p(x) \leq f_n(x)$ so

$$\int_{x\in X} g_m(x)\, d\mu(x) \leq \int_{x\in X} f_n(x)\, d\mu(x).$$

This holds for all $n \geq m$ so

$$\int_{x\in X} g_m(x)\, d\mu(x) \leq \inf_{n\geq m} \int_{x\in X} f_n(x)\, d\mu(x)$$

and

$$\sup_{m\in\mathrm{N}} \int_{x\in X} g_m(x)\, d\mu(x) \leq \sup_{m\in\mathrm{N}} \inf_{n\geq m} \int_{x\in X} f_n(x)\, d\mu(x).$$

Fatou's Lemma (3/3)

Combining

$$\sup_{m \in \mathbb{N}} \int_{x \in X} g_m(x)\, d\mu(x) \leq \sup_{m \in \mathbb{N}} \inf_{n \geq m} \int_{x \in X} f_n(x)\, d\mu(x)$$

with the equation

$$\int_{x \in X} \sup_{x \in X} g_m(x)\, d\mu(x) = \sup_{m \in \mathbb{N}} \int_{x \in X} g_m(x)\, d\mu(x)$$

obtained earlier, we find that

$$\int_{x \in X} \sup_{m \in \mathbb{N}} g_m(x)\, d\mu(x) \leq \sup_{m \in \mathbb{N}} \inf_{n \geq m} \int_{x \in X} f_n(x)\, d\mu(x),$$

or, in view of how g was defined,

$$\int_{x \in X} \sup_{m \in \mathbb{N}} \inf_{n \geq m} f_n(x)\, d\mu(x) \leq \sup_{m \in \mathbb{N}} \inf_{n \geq m} \int_{x \in X} f_n(x)\, d\mu(x).$$

Dominated Convergence Theorem (1/4)

Suppose $(X, \mathcal{B}, \mu)$ *is a measure space and* $f\colon \mathbf{N} \times X \to \mathbf{R}$ *is a function and* $g\colon X \to [0, +\infty]$ *is a function such that*

$$\lim_{m\to\infty} f_m(x)$$

exists for all $x \in X$*,* $f_m(x)$ *is integrable as a functions of* x *for each* $m \in \mathbf{N}$*,*

$$\int_{x\in X} g(x)\, d\mu(x) < +\infty$$

and

$$|f_m(x)| \leq g(x)$$

for all $m \in \mathbf{N}$*. Then*

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) = \int_{x\in X} \lim_{m\to\infty} f_m(x)\, d\mu(x).$$

Dominated Convergence Theorem (2/4)

Define

$$h_m(x) = g(x) + f_m(x).$$

Then $h_m(x) \geq 0$ for all $m \in \mathbf{N}$ and $x \in X$. By Fatou's Lemma,

$$\int_{x \in X} \sup_{m \in \mathrm{N}} \inf_{n \geq m} h_n(x)\, d\mu(x) \leq \sup_{a \in \mathrm{N}} \inf_{n \geq m} \int_{x \in X} h_n(x)\, d\mu(x).$$

Now

$$\begin{aligned} \sup_{m \in \mathrm{N}} \inf_{n \geq m} h_n(x) &= g(x) + \sup_{m \in \mathrm{N}} \inf_{n \geq m} f_n(x) \\ &= g(x) + \lim_{m \to \infty} f_m(x). \end{aligned}$$

Also,

$$\int_{x \in X} h_m(x)\, d\mu(x) = \int_{x \in X} g(x)\, d\mu(x) + \int_{x \in X} f_m(x)\, d\mu(x)$$

Dominated Convergence Theorem (3/4)

$$\sup_{m\in\mathrm{N}} \inf_{n\geq m} \int_{x\in X} h_n(x)\, d\mu(x) = \int_{x\in X} g(x)\, d\mu(x) + \sup_{m\in\mathrm{N}} \inf_{n\geq m} \int_{x\in X} f_n(x)\, d\mu(x).$$

Therefore

$$\int_{x\in X} g(x)\, d\mu(x) + \int_{x\in X} \lim_{m\to\infty} f_m(x)\, d\mu(x) \leq \int_{x\in X} g(x)\, d\mu(x) + \sup_{m\in\mathrm{N}} \inf_{n\geq m} \int_{x\in X} f_n(x)\, d\mu(x)$$

Because $\int_{x\in X} g(x)\, d\mu(x) < +\infty$ we can cancel it to get

$$\int_{x\in X} \lim_{m\to\infty} f_m(x)\, d\mu(x) \leq \sup_{m\in\mathrm{N}} \inf_{n\geq m} \int_{x\in X} f_n(x)\, d\mu(x)$$

Dominated Convergence Theorem (4/4)

We can apply the same argument with $-f_m(x)$ in place of $f_m(x)$ to get

$$\int_{x\in X} \lim_{m\to\infty} -f_m(x)\,d\mu(x) \leq \sup_{m\in\mathrm{N}} \inf_{n\geq m} \int_{x\in X} -f_m(x)\,d\mu(x),$$

or, equivalently,

$$\inf_{m\in\mathrm{N}} \sup_{n\geq m} \int_{x\in X} f_m(x)\,d\mu(x) \leq \int_{x\in X} \lim_{m\to\infty} f_m(x).$$

It follows that

$$\inf_{m\in\mathrm{N}} \sup_{n\geq m} \int_{x\in X} f_n(x)\,d\mu(x) \leq \sup_{m\in\mathrm{N}} \inf_{n\geq m} \int_{x\in X} f_n(x)\,d\mu(x)$$

and therefore $\lim_{m\to\infty} \int_{x\in X} f_m(x)\,d\mu(x)$ exists and is equal to their common value. So

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\,d\mu(x) = \int_{x\in X} \lim_{m\to\infty} f_m(x)\,d\mu(x).$$