MAU22200 Lecture 50

John Stalker

Trinity College Dublin

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Upper and lower integrals

Last time we found an alternate characterisation of integrability.

Suppose $(X, \mathcal{B}, \mu)$ is a content/measure space, $Y \subseteq [-\infty, +\infty]$ and $g \colon X \to Y$. Then

$$\underline{\int}_{x \in X} g(x)\, d\mu(x) \le \overline{\int}_{x \in X} g(x)\, d\mu(x).$$

g is integrable with respect to $(X, \mathcal{B}, \mu)$ if and only if also

$$\overline{\int}_{x \in X} g(x)\, d\mu(x) \le \underline{\int}_{x \in X} g(x)\, d\mu(x)$$

and both the upper and lower integrals belong to Y. In that case the integral of g is their common value.

We'll use this in a variety of ways.

Completions (1/3)

One use of the upper and lower integral criterion for integrability is to compare integrability for a content/measure space and its completion.

> *Suppose* $(X, \mathcal{B}, \mu)$ *is a content/measure space and* $(X, \mathcal{B}^\dagger, \mu^\dagger)$ *is its completion. Suppose* $Y \subseteq [-\infty, +\infty]$*. Then* $g\colon X \to Y$ *is integrable with respect to* $(X, \mathcal{B}^\dagger, \mu^\dagger)$ *if and only if it is integrable with respect to* $(X, \mathcal{B}, \mu)$*. The two integrals are then equal.*

Half of this isn't really new. $(X, \mathcal{B}^\dagger, \mu^\dagger)$ is a refinement of $(X, \mathcal{B}, \mu)$, so g is integrable with respect to $(X, \mathcal{B}^\dagger, \mu^\dagger)$ if it is integrable with respect to $(X, \mathcal{B}, \mu)$. The integrals are equal in this case. This was a theorem from Lecture 48. The new part is that g is integrable with respect to $(X, \mathcal{B}, \mu)$ if it is integrable with respect to $(X, \mathcal{B}^\dagger, \mu^\dagger)$. This direction doesn't work for refinements in general, only for completions.

Completions (2/3)

The idea of the proof is that simple/semisimple functions with respect $(X, \mathcal{B}^\dagger)$ may not be simple/semisimple functions with respect to $(X, \mathcal{B})$, but they can be approximated arbitrarily closely from below or from above by them. The approximation is in the sense that the integrals can be made arbitrarily close. It then follows that the upper and lower integrals with respect to $(X, \mathcal{B}^\dagger)$ are the same as those with respect to $(X, \mathcal{B})$. From the theorem then the integrals are also the same.

The content space version is easier, since simple functions are linear combinations of characteristic functions: $g = \sum_{i=1}^{m} c_i \chi_{E_i}$. For a single characteristic function χ_F where $F \in \mathcal{B}^\dagger$ we can find, for any $\epsilon > 0$, sets $D, H \in \mathcal{B}$ such that $F \triangle H \subseteq D$ and $\mu(D) < \epsilon$. Let $E = H \setminus D$ and $G = H \cup D$. Then $E \subseteq F \subseteq G$ so $\chi_E(x) \leq \chi_F(x) \leq \chi_G(x)$.

Completions (3/3)

If $E = H \setminus D$ and $G = H \cup D$ then $G = E \cup D$ so $\mu(G) \leq \mu(E) + \mu(D) < \mu(E) + \epsilon$. So

$$\int_{x \in X} \chi_F(x)\, d\mu^\dagger(x) = \mu^\dagger(F) < \mu(E) + \epsilon = \int_{x \in X} \chi_E(x)\, d\mu(x) + \epsilon$$
$$\int_{x \in X} \chi_F(x)\, d\mu^\dagger(x) = \mu^\dagger(F) > \mu(G) + \epsilon = \int_{x \in X} \chi_G(x)\, d\mu(x) - \epsilon$$

We can extend this to linear combinations. There are some minor complications due to working in a subset of $[-\infty, +\infty]$ rather than $\mathbf{R}$, and from the presence of sets $E \in \mathcal{Q}$ with $\mu(E) = +\infty$ if $\mu(X) = +\infty$.

There are some extra complications when we move from content spaces to measure spaces. For example, every semisimple function is a countable linear combination of characteristic functions, but not every countable linear combination of characteristic functions is semisimple! I won't give the details here.

The Riemann integral

The *Riemann integral* of a function $f : \mathbf{R}$ to $\mathbf{R}$ is defined to be its integral with respect to the content space $(\mathbf{R}, \mathcal{I}, \mu_{\mathcal{I}})$.
The preceding theorem shows that we'd get the same thing from $(\mathbf{R}, \mathcal{J}, \mu_{\mathcal{J}})$ though. This may not look like the familiar Riemann integral but the upper and lower integrals are just the upper and lower Darboux integrals, so it's the same.
At this point almost all of the properties of the Riemann integral are consequences of things we've prove in more generality, e.g. monotonicity, linearity, etc. There's one exception though!

Compactly supported continuous functions (1/2)

> *All compactly supported continuous functions are Riemann integrable.*

What does compactly supported mean? The *support* of g is the set

$$\overline{g^*(\mathbf{R} \setminus \{0\})}.$$

x belongs to the support of g if and only if it has no neighbourhood in which g is identically zero. g is said to be *compactly supported* if its support is compact. This makes sense for real valued functions on any topological space.
Note that plenty of functions are Riemann integrable without being continuous, e.g. χ_C where C is the Cantor set is discontinuous at every point in C, so at uncountably many points, but is Riemann integrable! It turns out that all Riemann integrable functions are (essentially) compactly supported though. Not every compactly supported function is Riemann integrable. $\chi_{[0,1]\cap\mathbf{Q}}$ is compactly supported but not Riemann integrable.

Compactly supported continuous functions (2/2)

All compactly supported continuous functions are Riemann integrable.

Why? We can approximate such functions arbitrarily well by simple functions with respect to $(\mathbf{R}, \mathcal{I})$. The support of g is compact, so bounded, so contained in an interval $[-r, r]$. We take a partition $\mathcal{Q}$ with $(-\infty, -r)$, $(r, +\infty)$ and $2n$ intervals of length r/n in $[-r, r]$. Define $f, h \colon \mathbf{R} \to \mathbf{R}$ by $f(x) = \inf_{y \in E} g(y)$ and $h(x) = \sup_{z \in E} g(z)$, where E is the unique element of $\mathcal{Q}$ containing x. The lower integral is at least $\int_{x \in \mathrm{R}} f(x)\, d\mu(x)$ and the upper integral is at most $\int_{x \in \mathrm{R}} h(x)\, d\mu(x)$.

$$\int_{x \in \mathrm{R}} h(x)\, d\mu(x) \leq \int_{x \in \mathrm{R}} f(x)\, d\mu(x) + 2nr \sup_{E \in \mathcal{Q}} \sup_{y, z \in E} (g(z) - g(y)).$$

g is continuous on the compact set $[-r, r]$ so is uniformly continuous, so we can make $\sup_{E \in \mathcal{Q}} \sup_{y, z \in E} (g(z) - g(y))$ as small as we want by choosing n large enough.

Measurable functions (1/2)

Suppose $(X, \mathcal{B})$ is a measurable space, i.e. that $\mathcal{B}$ is a σ-algebra on X. Suppose $(Y, \mathcal{T})$ is topological space (but really we only care about $Y = [0, +\infty]$ or $Y = \mathbf{R}^n$, usually with $n = 1$). Then $g \colon X \to Y$ is called *measurable* if $\mathcal{B}_Y \subseteq g^{**}(\mathcal{B})$ where $\mathcal{B}_Y$ is the Borel σ-algebra on Y. Equivalently, g is measurable if and only if $f^*(E) \in \mathcal{B}$ for every Borel set E in Y.

- If f is continuous and g is measurable then $f \circ g$ is measurable.
- If f is measurable then $|f|$ is also measurable.
- (Finite) linear combinations of measurable functions are measurable.
- (Finite) products of measurable functions are measurable.
- If f is a sequence of measurable functions then $\sup_{n \in \mathrm{N}} f_n$, $\inf_{n \in \mathrm{N}} f_n$ and $\lim_{n \to \infty} f_n$ are all measurable, assuming they exist.

Measurable functions (2/2)

We saw with real valued sums that summability is equivalent to absolute summability. This is definitely not true for integrals, but it is true if we restrict our attention to measurable functions with respect to the completion of a measure space. As before, one direction follows from a version of the comparison test:

> *Suppose $(X, \mathcal{B}, \mu)$ is a measure space. Suppose $f, g \colon X \to \mathbf{R}$ are measurable functions such that $|f(x)| \leq g(x)$ for almost all $x \in X$. If*
>
> $$\int_{x \in X} g(x)\, d\mu(x) < +\infty$$
>
> *then f is integrable.*

That absolute integrability implies integrability for measurable function is the special case $f = g$. There are counter-examples if we drop the measurability assumption. The reverse direction is harder.