MAU22200 Lecture 49

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Almost everywhere

A statement is said to hold for *almost all* points in a content/measure space if there is a set of content/measure zero in the complement of which it holds.
For an uninteresting example, a statement holds for almost all x with respect to counting measure if and only if it holds for all x.
For a more interesting example, for almost all $x \in \mathbf{R}$ there is a 7 somewhere in the decimal expansion of x. The content here can be taken to be Jordan content. The proof of this closely follows the proof that the Cantor set has content zero, since the Cantor set is the set of points in $[0, 1]$ which have no 1 in their trinary expansion.
Still sticking to Jordan content, it's not true that almost every real number is irrational. This will be true of Lebesgue measure, which we will see after the break.

Functions which agree almost everywhere (1/3)

Suppose $(X, \mathcal{B}, \mu)$ is a content/measure space, $Y \subseteq [-\infty, +\infty]$, and $g, h\colon X \to Y$ are such that $g(x) = h(x)$ for almost all $x \in X$. If g integrable then so is h and

$$\int_{x \in X} g(x)\, d\mu(x) = \int_{x \in X} h(x)\, d\mu(x).$$

The hypothesis that $g(x) = h(x)$ almost everywhere means that there is an $E \in \mathcal{B}$ with $\mu(E) = 0$ such that $g(x) = h(x)$ for all $x \in X \setminus E$. With notation as in the definition of the integral, suppose $T \in R_g^{**}(\mathcal{E})$, i.e. that $R_g^*(T) \in \mathcal{E}$. Let $V = R_g^*(T)$. Then $V \in \mathcal{E}$ so there is $\mathcal{Q} \in \mathbf{P}$ such that $\alpha(\mathcal{Q}) \subseteq V$. Let $\mathcal{R}$ be the common refinement of $\mathcal{Q}$ and $\{E, X \setminus E\}$. If $w \in \alpha(\mathcal{R})$ then $w \in \alpha(\mathcal{Q})$ and so $w \in V$. So $\alpha(\mathcal{R}) \subseteq V$. Also $\alpha(\mathcal{R}) \in \mathcal{E}$. For $w \in \alpha(\mathcal{R})$ we have $E \in \mathcal{B}_{\mathcal{R}}$ so $\sum_{x \in E} w(x) = \mu(E) = 0$ and $w(x) = 0$ for all $x \in E$ and hence $\sum_{x \in E} g(x)w(x) = 0 = \sum_{x \in E} h(x)w(x)$.

Functions which agree almost everywhere (2/3)

On the other hand, we have $g(x) = h(x)$ for $x \in X \setminus E$ so

$$\sum_{x \in X \setminus E} g(x)w(x) = \sum_{x \in X \setminus E} h(x)w(x).$$

Then

$$\begin{aligned}
R_g(w) &= \sum_{x \in X} g(x)w(x) \\
&= \sum_{x \in E} g(x)w(x) + \sum_{x \in X \setminus E} g(x)w(x) \\
&= \sum_{x \in E} h(x)w(x) + \sum_{x \in X \setminus E} h(x)w(x) \\
&= \sum_{x \in X} h(x)w(x) = R_h(w).
\end{aligned}$$

So $R_g(w) = R_h(w)$ for all $w \in \alpha(\mathcal{R})$.

Functions which agree almost everywhere (3/3)

Let $S = R_{h*}(\alpha(\mathcal{R}))$. If $w \in \alpha(\mathcal{R})$ then $R_h(w) \in S$ so $w \in R_h^*(S)$. Therefore $\alpha(\mathcal{R}) \subseteq R_h^*(S)$. $\alpha(\mathcal{R}) \in \mathcal{E}$ and $\mathcal{E}$ is upward closed so $R_h^*(S) \in \mathcal{E}$ and hence $S \in R_h^{**}(\mathcal{E})$. If $z \in S$ then $z = R_h(w)$ for some $w \in \alpha(\mathcal{R})$. Then $z = R_g(w)$. $\alpha(\mathcal{R}) \subseteq V$ so $w \in V$ and hence $z = R_g(w) \in T$, since $V = R_g^*(T)$. This holds for all $z \in S$ so $S \subseteq T$. From this and $S \in R_h^{**}(\mathcal{E})$ it follows that $T \in R_h^{**}(\mathcal{E})$, since $R_h^{**}(\mathcal{E})$ is upward closed. T was an arbitrary element of $R_g^{**}(\mathcal{E})$ so $R_g^{**}(\mathcal{E}) \subseteq R_h^{**}(\mathcal{E})$. The same argument with the roles of g and h reversed gives $R_h^{**}(\mathcal{E}) \subseteq R_g^{**}(\mathcal{E})$, so

$$R_g^{**}(\mathcal{E}) = R_h^{**}(\mathcal{E}).$$

Therefore $R_g^{**}(\mathcal{E})$ converges if and only if $R_h^{**}(\mathcal{E})$ converges, in which case the limits are the same. In terms of integrals this means that g is integrable if and only if h is, in which case

$$\int_{x \in X} g(x)\, d\mu(x) = \int_{x \in X} h(x)\, d\mu(x).$$

Lower and upper integrals (1/2)

Suppose $(X, \mathcal{B}, \mu)$ is a content/measure space, $Y \subseteq [-\infty, +\infty]$ and $g\colon X \to Y$ is a function.

The *lower integral* of g with respect to $(X, \mathcal{B}, \mu)$,

$$\underline{\int}_{x\in X} g(x)\, d\mu(x),$$

is the supremum of all integrals $\int_{x\in X} f(x)\, d\mu(x)$ where f ranges over the simple/semisimple functions such that $f(x) \leq g(x)$ for almost all $x \in X$.

The *upper integral* of g with respect to $(X, \mathcal{B}, \mu)$,

$$\overline{\int}_{x\in X} g(x)\, d\mu(x),$$

is the infimum of all integrals $\int_{x\in X} h(x)\, d\mu(x)$ where h ranges over the simple/semisimple functions such that $g(x) \leq h(x)$ for almost all $x \in X$.

Lower and upper integrals (2/2)

The "for almost all $x \in X$" is natural in since we only really care about the integrals. It's also necessary to make the following theorem work:

Suppose $(X, \mathcal{B}, \mu)$ is a content/measure space, $Y \subseteq [-\infty, +\infty]$ and $g \colon X \to Y$. Then

$$\underline{\int}_{x \in X} g(x)\, d\mu(x) \leq \overline{\int}_{x \in X} g(x)\, d\mu(x).$$

g is integrable with respect to $(X, \mathcal{B}, \mu)$ if and only if also

$$\overline{\int}_{x \in X} g(x)\, d\mu(x) \leq \underline{\int}_{x \in X} g(x)\, d\mu(x)$$

and both the upper and lower integrals belong to Y. In that case the integral of g is their common value.

This criterion is easier to work with than the definition.

Sketch of the proof (1/3)

The proof is made up of two parts, one general and one specific to integration.

Suppose $\mathcal{E}$ is a filter on a set X, $Y \subseteq [-\infty, +\infty]$, and $r\colon X \to Y$ is a function. Then

$$\sup_{V \in \mathcal{E}} \inf_{w \in V} r(w) \le \inf_{V \in \mathcal{E}} \sup_{w \in V} r(w)$$

in $[-\infty, +\infty]$. $r^{**}(\mathcal{E})$ is convergent in Y if and only if

$$\inf_{V \in \mathcal{E}} \sup_{w \in V} r(w) \in Y,$$

$$\sup_{V \in \mathcal{E}} \inf_{w \in V} r(w) \in Y,$$

and

$$\inf_{V \in \mathcal{E}} \sup_{w \in V} r(w) \le \sup_{V \in \mathcal{E}} \inf_{w \in V} r(w).$$

In this case their common value is the limit of $r^{**}(\mathcal{E})$.

Sketch of the proof (2/3)

The convergence criterion on the preceding slide is the filter version of one we've already seen for sequences and nets. We apply it to $r = R_g$ and $\mathcal{E}$ as in the definition of the integral. To do that we need to identify $\inf_{V\in\mathcal{E}} \sup_{w\in V} R_g(w)$ and $\sup_{V\in\mathcal{E}} \inf_{w\in V} R_g(w)$.

Suppose that $(X, \mathcal{B}, \mu)$ is a content/measure space, $Y \subseteq [-\infty, +\infty]$ and $g\colon X \to Y$ is a function. Then

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} R_g(w) = \overline{\int}_{x\in X} g(x)\, d\mu(x)$$

and

$$\sup_{V\in\mathcal{E}} \inf_{w\in V} R_g(w) = \underline{\int}_{x\in X} g(x)\, d\mu(x).$$

Sketch of the proof (3/3)

Suppose f is a simple/semisimple function such that $f(x) \leq g(x)$ for almost all $x \in X$, i.e. there's a finite/countable partition $\mathcal{Q}$ of x such that f is constant on each element of $\mathcal{Q}$. If $(X, \mathcal{B}, \mu)$, $\mathcal{Q}$ and w are compatible then

$$\int_{x \in X} f(x)\, d\mu(x) = \sum_{x \in X} f(x) w(x) \leq \sum_{x \in X} g(x) w(x) = R_g(w).$$

This is not quite true, but it's close enough. The heart of the proof is showing that for any $\mathcal{Q}$ we can choose f and w to get $\int_{x \in X} f(x)\, d\mu(x)$ and $R_g(w)$ arbitrarily close to equal.
It's rather complicated because there are many special cases to be considered or, preferably, avoided. For details see the notes.