MAU22200 Lecture 48

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Compatibility (1/3)

$(X, \mathcal{B}, \mu)$ is called a *content/measure space* if $\mathcal{B}$ is a Boolean/σ-algebra on X and μ is a content/measure on $(X, \mathcal{B})$. The convention here that that you consistently take either the left or right side of the / and you get something sensible.

A *partition* on X is a $\mathcal{Q} \in \wp(\wp(X))$ such that $\varnothing \notin \mathcal{Q}$ and if $E, F \in \mathcal{Q}$ then $E = F$ or $E \cap F = \varnothing$.

A *system of weights* on X is just a function $w \colon X \to [0, +\infty]$.

A content/measure space $(X, \mathcal{B}, \mu)$ a partition $\mathcal{P}$ and a system of weights $w \colon X \to [0, +\infty]$ are called *compatible* if $\mathcal{P} \subseteq \mathcal{B}$ and $\mu(E) = \sum_{x \in E} w(x)$ for all $E \in \mathcal{P}$.

How does this depend on the choices of $(X, \mathcal{B}, \mu)$ and $\mathcal{P}$?

> *If $(X, \mathcal{B}', \mu')$ is a refinement of $(X, \mathcal{B}, \mu)$ then $(X, \mathcal{B}', \mu')$, $\mathcal{P}$ and w are compatible if $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w are. $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w are compatible if $(X, \mathcal{B}', \mu')$, $\mathcal{P}$ and w are and $\mathcal{P} \subseteq \mathcal{B}$.*

Compatibility (2/3)

Suppose $(X, \mathcal{B}, \mu)$ is a content/measure space and $\mathcal{P}$ and $\mathcal{Q}$ are finite/countable partitions of X and w is a system of weights on X. If $\mathcal{Q}$ is a refinement of $\mathcal{P}$ and $(X, \mathcal{B}, \mu)$, $\mathcal{Q}$ and w are compatible then so are $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w.

This is a consequence of the following characterisation of compatibility in terms of refinements:

Suppose $(X, \mathcal{B}, \mu)$ is a content/measure space, $\mathcal{P}$ is a finite/countable partition of X and w is a system of weights on X. Define $\mu_w(E) = \sum_{x \in E} w(x)$ for $E \in \wp(X)$ and $\mu_{\mathcal{P}}(E) = \sum_{x \in E} w(x)$ for $E \in \mathcal{B}_{\mathcal{P}}$ where $\mathcal{B}_{\mathcal{P}}$ is the atomic algebra associated to $\mathcal{P}$. Then $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w are compatible if and only if both $(X, \mathcal{B}, \mu)$ and $(X, \wp(X), \mu_w)$ are refinements of $(X, \mathcal{B}_{\mathcal{P}}, \mu_{\mathcal{P}})$.

Combining this with the fact the the refinement of a refinement is a refinement gives the proposition above.

Compatibility (3/3)

Here's a diagram proof:

$$\begin{tikzcd}
(X, \mathcal{B}, \mu) \arrow[dr, red] \arrow[drr, dashed] & & \\
& (X, \mathcal{B}_{\mathcal{Q}}, \mu_{\mathcal{Q}}) \arrow[r, rightsquigarrow, blue] & (X, \mathcal{B}_{\mathcal{P}}, \mu_{\mathcal{P}}) \\
(X, \wp(X), \mu_w) \arrow[ur, red] \arrow[urr, dashed] & &
\end{tikzcd}$$

Arrows in the diagram mean "is a refinement of".
The red solid arrows exist because $(X, \mathcal{B}_{\mathcal{Q}}, \mu_{\mathcal{Q}})$ is a refinement of $(X, \mathcal{B}, \mu)$ and of $(X, \wp(X), \mu_w)$, i.e. because $(X, \mathcal{B}, \mu)$, $\mathcal{Q}$ and w are compatible. The blue squiggly arrow exists because $(X, \mathcal{B}_{\mathcal{Q}}, \mu_{\mathcal{Q}})$ is a refinement of $(X, \mathcal{B}_{\mathcal{P}}, \mu_{\mathcal{P}})$, i.e. because $\mathcal{Q}$ is a refinement of $\mathcal{P}$, The black dashed arrows exist because the refinement of a refinement is a refinement. Because they exist, $(X, \mathcal{B}_{\mathcal{P}}, \mu_{\mathcal{P}})$ is a refinement of $(X, \mathcal{B}, \mu)$ and of $(X, \wp(X), \mu_w)$, so $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w are compatible.

Definition of the integral

Suppose $(X, \mathcal{B}, \mu)$ is a content/measure space, $Y = [0, +\infty]$ or $Y = \mathbf{R}$, and $f \colon X \to Y$ is a function.

Let $\mathbf{P}$ be the set of finite/countable subsets of $\mathcal{B}$ which are partitions of X. Let U be the set of systems of weights w such that $\sum_{x \in X} w(x) f(x)$ converges (in Y). This is all of them if $Y = [0, +\infty]$, but generally not if $Y = \mathbf{R}$. Define $R \colon U \to Y$ by $R(w) = \sum_{x \in X} w(x) f(x)$. Define $\alpha \colon \mathbf{P} \to \wp(U)$ by saying $w \in \alpha(\mathcal{Q})$ if and only if $(X, \mathcal{B}, \mu)$, $\mathcal{Q}$ and w are compatible. Let $\mathcal{E}$ be the upward closure of $\alpha_*(\mathbf{P})$.

f is said to be *integrable* with respect to $(X, \mathcal{B}, \mu)$ if $\varnothing \notin \mathcal{E}$ and $R^{**}(\mathcal{E})$ is convergent. In that case we call its limit the *integral* of f with respect to $(X, \mathcal{B}, \mu)$, and write it as $\int_{x \in X} f(x) \, d\mu(x)$.

$\mathbf{P}$ is a non-empty directed set, α is monotone, and $\mathcal{E}$ is filter on X. Also, Y is Hausdorff, so there is at most one element of Y to which $R^{**}(\mathcal{E})$ converges.

Elementary properties

No definition of integrals is easy to use on examples, and this one is no exception!

It does fit into the framework from Section 1.16 though, so we get some theorems for free. We already got one, the uniqueness of the integral. Here's another:

> *Suppose* $(X, \mathcal{B}, \mu)$ *is a content/measure space,* f, g *are integrable, and* $f(x) \leq g(x)$ *for all* $x \in X$*. Then* $\int_{x \in X} f(x)\, d\mu(x) \leq \int_{x \in X} g(x)\, d\mu(x)$*.*

And another:

> *Suppose* $(X, \mathcal{B}, \mu)$ *is a content/measure space* $c_1, \ldots, c_m \in Y$ *and* $f_1, \ldots, f_m$ *are integrable with respect to* $(X, \mathcal{B}, \mu)$*. Define* g *by* $g(x) = \sum_{i=1}^{m} c_i f_i(x)$*. Then* g *is integrable and* $\int_{x \in X} g(x)\, d\mu(x) = \sum_{i=1}^{m} c_i \int_{x \in X} f_i(x)\, d\mu(x)$*.*

A less elementary property

Suppose $(X, \mathcal{B}, \mu)$ is a content/measure space and $(X, \mathcal{B}', \mu')$ is a content/measure space which is a refinement of $(X, \mathcal{B}, \mu)$. If f is integrable with respect to $(X, \mathcal{B}, \mu)$ then it is also integrable with respect to $(X, \mathcal{B}', \mu')$ and

$$\int_{x \in X} f(x)\, d\mu'(x) = \int_{x \in X} f(x)\, d\mu(x).$$

In addition to being true if both are content spaces or if both are measure spaces this one holds if $(X, \mathcal{B}, \mu)$ is a content space and $(X, \mathcal{B}', \mu')$ is a measure space. It can fail though if $(X, \mathcal{B}, \mu)$ is a measure space and $(X, \mathcal{B}', \mu')$ is a content space.

This one's also a consequence of properties of limits,

Lemma 4.6.2a:

If $(X, \mathcal{T})$ is a topological space, $\mathcal{F}$ is a convergent filter on X and $\mathcal{G}$ is a filter on X such that $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G}$ is a convergent filter.

Simple/semisimple functions (1/2)

Can we actually evaluate any integrals?

> *Suppose $\mathcal{B}$ is a Boolean/σ-algebra and Y is a set. $f\colon X \to Y$ is called simple/semisimple if there is a finite/countable partition $\mathcal{Q}$ of X such that $\wp(Y) \subseteq f^{**}(\mathcal{B})$.*

This may seem opaque, but simple/semisimple functions are really simple. f is simple/semisimple if and only if there is a finite/countable partition $\mathcal{Q}$ of X and a function $\varphi\colon \mathcal{Q} \to Y$ such that $f(x) = \varphi(E)$ where x is the unique element of $\mathcal{Q}$ such that $x \in E$. The easiest example of a simple function is a *characteristic function*,

$$\chi_F(x) = \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{if } x \notin F. \end{cases}$$

All finite sets are countable so every simple function is semisimple.

Simple/semisimple functions (2/2)

The integral of a simple or semisimple function is

$$\int_{x\in X} f(x)\,d\mu(x) = \sum_{E\in\mathcal{Q}} \varphi(E)\mu(E).$$

In particular,

$$\int_{x\in X} \chi_F(x)\,d\mu(x) = \mu(F).$$

Every simple/semisimple function is a finite/countable linear combination of characteristic functions.
In the next lecture we'll see that integrable functions are those which can be well approximated by simple/semisimple functions.