MAU22200 Lecture 47

John Stalker

Trinity College Dublin

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Morphisms

Structures often come with an appropriate notion of structure-preserving function, e.g.

- vector spaces and linear transformations,
- groups and group homomorphisms,
- topological spaces and continuous functions, (this one's a little subtle, because the same underlying set can have different topologies and the identity function might not be continuous!)
- normed vector spaces and bounded linear transformations

In each case there's an appropriate notion of an identity and of composition. There's a whole subject, Category Theory, about what you can conclude with no other information about your objects. For measure spaces or content spaces we have a notion of a *morphism* of measure or content spaces.

Morphisms of content spaces (1/2)

Suppose $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ are content spaces (or measure spaces) and $f : X \to Y$ is a function. f is called a *morphism* if

- $\mathcal{C} \subseteq f^{**}(\mathcal{B})$
- $\nu(E) = \mu(f^*(E))$ whenever both sides make sense, i.e. when $E \in \mathcal{C}$.

In some situations we need to distinguish measure and content spaces, but here we don't, since every measure space is a content space.
The appearance of f^{**} shouldn't be too surprising. We've seen it before when discussing continuous functions.
Also, we saw that if $\mathcal{B}$ is a Boolean algebra then so is $f^{**}(\mathcal{B})$ and if $\mathcal{B}$ is a σ-algebra then so is $f^{**}(\mathcal{B})$.

Morphisms of content spaces (2/2)

Compositions behave as they should, i.e. if $(X, \mathcal{B}, \mu)$, $(Y, \mathcal{C}, \nu)$ and $(Z, \mathcal{D}, \xi)$ are content spaces (or measure spaces) and $f : X \to Y$ and $g : Y \to Z$ are morphisms then so is $g \circ f$. Similar to topological spaces and continuous functions, we can have different content space structures on the same underlying set, and the set-theoretic identity function between the underlying sets may or may not be a morphism. For example $i : \mathbf{R} \to \mathbf{R}$, defined by $i(x) = x$, is a morphism from $(\mathbf{R}, \mathcal{J}, \mu_{\mathcal{J}})$ to $(\mathbf{R}, \mathcal{I}, \mu_{\mathcal{I}})$, but not from $(\mathbf{R}, \mathcal{I}, \mu_{\mathcal{J}})$ to $(\mathbf{R}, \mathcal{J}, \mu_{\mathcal{I}})$. In topology we mostly consider continuous functions between different sets, and consider only one topology on each set. In measure theory we mostly consider a single set with multiple Boolean algebras and contents, some of which may be σ-algebras and measures. The definitions are flexible enough to deal with the less common cases though.

Refinements (1/2)

$(X, \mathcal{B}', \mu')$ is called a *refinement* of $(X, \mathcal{B}, \mu)$ if $\mathcal{B} \subseteq \mathcal{B}'$ and $\mu'(E) = \mu(E)$ for all $E \in \mathcal{B}$, or, equivalently, if the identity function $i \colon X \to X$, i.e. $i(x) = x$, is a morphism from $(X, \mathcal{B}', \mu')$ to $(X, \mathcal{B}, \mu)$.

Showing these are equivalent is straightforward, and is done in the notes.

Example: $(\mathbf{R}, \mathcal{J}, \mu_{\mathcal{J}})$ is a refinement of $(\mathbf{R}, \mathcal{I}, \mu_{\mathcal{I}})$. More generally, if $(X, \mathcal{B}^\dagger, \mu^\dagger)$ is the completion of $(X, \mathcal{B}, \mu)$ then $(X, \mathcal{B}^\dagger, \mu^\dagger)$ is a refinement of $(X, \mathcal{B}^\dagger, \mu^\dagger)$. There are (sort of) two different completions, for content spaces and for measure spaces, and the statement above holds for either.

Why are these called refinements? What's the connection with refinement of partitions?

Refinements (2/2)

Suppose $\mathcal{P}$, $\mathcal{Q}$ are partitions of X and $\mathcal{B}$ and $\mathcal{C}$ are the associated atomic algebras, i.e. $\mathcal{B}$ is the set of unions of elements of $\mathcal{P}$ and $\mathcal{C}$ is the set of unions of elements of $\mathcal{Q}$. For any content ν on $(X, \mathcal{C})$ we can define a content μ on $(X, \mathcal{B})$ by $\nu(E) = \mu(E)$.
$\mathcal{B} \subseteq \mathcal{C}$, so $(X, \mathcal{C}, \nu)$ is a refinement of $(X, \mathcal{B}, \mu)$.
Conversely, suppose $(X, \mathcal{C}, \nu)$ is a refinement of $(X, \mathcal{B}, \mu)$. Then $\mathcal{Q}$ is a refinement of $\mathcal{P}$.
If $(X, \mathcal{C}, \nu)$ is a refinement of $(X, \mathcal{B}, \mu)$ and $(X, \mathcal{D}, \xi)$ is a refinement of $(X, \mathcal{C}, \nu)$ then $(X, \mathcal{D}, \xi)$ is a refinement of $(X, \mathcal{B}, \mu)$. This is a consequence of our proposition about compositions of morphisms.
In the special case where $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ come from partitions it follows from the fact that the refinement of a refinement of partitions is a refinement.
As with partitions, content spaces (or measure spaces) on a given set form a non-empty directed set.

Integrals (1/2)

I reviewed Riemann integration in Section 1.20 and Lecture 9.

The idea was to introduce finite partitions of an interval into intervals and Riemann sums for each such partition, then take a limit in our generalised definition of limits. Then the properties of limits, e.g. uniqueness, monotonicity and linearity, imply the corresponding properties for Riemann integrals. You can rephrase the bits about integrals in terms of the Boolean algebra $\mathcal{I}$.

We can now generalise this construction. There are actually four slightly different generalisations, since we have two binary choice to make:

- We can use Boolean algebras, contents and finite partitions again or we can use σ-algebras, measures and countable partitions.
- We can look at functions with values in $[0, +\infty]$ (with better convergence and monotonicity results) or with values in $\mathbf{R}$ (with better linearity properties). This parallels what we did we did with sums.

Integrals (2/2)

We're mostly interested in σ-algebras and measures and functions with values in $\mathbf{R}$, but each version of the theory has its uses.

Until we start exchanging limits and integrals the results and arguments for the four cases are fairly similar. In some cases they're identical.

The σ-algebras and measures versions are better behaved for exchanging limits and integrals, which is why this module has a second semester.

We're still missing a crucial piece of the puzzle though: a measure analogous to, and a refinement of, Jordan content. That will be Lebesgue measure, and we'll construct it once we have our theory of integration.