MAU22200 Lecture 46

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Atomic algebra

Boolean/σ-algebras on a set X are those $\mathcal{B} \in \wp(\wp(X))$ such that

- $\varnothing \in \mathcal{B}$
- if $E \in \mathcal{B}$ then $X \setminus E \in \mathcal{B}$
- if $\mathcal{A}$ is a finite/countable subset of $\mathcal{B}$ then $\bigcup_{E \in \mathcal{A}} E \in \mathcal{B}$.

For σ-algebras this was a definition. For Boolean algebras we only required $E \cup F \in \mathcal{B}$ for $E, F \in \mathcal{B}$, but that's equivalent to what's above. What if we drop the size restriction on $\mathcal{A}$ entirely?

An *atomic algebra* on X is a $\mathcal{B} \in \wp(\wp(X))$ such that

- $\varnothing \in \mathcal{B}$
- if $E \in \mathcal{B}$ then $X \setminus E \in \mathcal{B}$
- if $\mathcal{A}$ is a subset of $\mathcal{B}$ then $\bigcup_{E \in \mathcal{A}} E \in \mathcal{B}$.

Just as every Boolean algebra is a σ-algebra, but not vice-versa, every σ-algebra is an atomic algebra, but not vice versa. The Borel algebra is a σ-algebra but is not an atomic algebra.

Partitions

A *partition* of a set X is a $\mathcal{P} \in \wp(\wp(X))$ such that $X = \bigcup_{E \in \mathcal{P}} E$ and for all $E, F \in \mathcal{P}$, $E \neq \varnothing$, $F \neq \varnothing$ and either $E = F$ or $E \cap F = \varnothing$. In other words, every $E \in \mathcal{P}$ is non-empty and $x \in X$ belongs to one and only one $E \in \mathcal{P}$. We've met these a few times already, without naming them. If $\mathcal{P}$ is a partition of a set S and $f \colon S \to [0, +\infty]$ is a function then

$$\sum_{s \in S} f(s) = \sum_{E \in \mathcal{P}} \sum_{s \in E} f(s).$$

This was Proposition 6.4.1 in the Chapter on sums.
If μ is a measure on a measurable space $(X, \mathcal{B})$ then

$$\mu(F) = \sum_{E \in \mathcal{P}} \mu(E),$$

provided $\mathcal{P}$ is a countable partition of F and $\mathcal{P} \subseteq \mathcal{B}$. This is essentially the second condition from the definition of a measure.

Equivalence relations

An equivalence relation on a set X is a binary relation $\sim$ such that for all $x, y, z \in X$ we have

- $x \sim x$,
- if $x \sim y$ then $y \sim x$,
- if $x \sim y$ and $y \sim z$ then $x \sim z$.

We discussed them in Chapter 2. The equivalence class of an element $x \in X$ is the set

$$\{y \in X : x \sim y\}.$$

Any two equivalence classes are either equal or disjoint.

These three things are sort of the same (1/3)

Atomic algebras, partitions and equivalence relations are three different points of view on the same concept. They aren't the same. For example $\{\{1\}, \{2, 3\}\}$ is a partition of $\{1, 2, 3\}$, but is not an atomic algebra, and certainly not an equivalence relation. Rather, given any one of the three there's a natural way to construct from it the other two.

For example, given an atomic algebra $\mathcal{B}$ we can get a partition by looking at its minimal non-empty elements, i.e. those which contain no other non-empty element as a subset. So

$$\mathcal{P} = \{F \in \mathcal{B} \setminus \{\varnothing\} \colon \forall E \in \mathcal{B} \setminus \{\varnothing\} \colon E \subseteq F \Rightarrow E = F\}.$$

For example, $\{\varnothing, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$ is an atomic algebra. Its minimal elements are the sets $\{1\}$ and $\{2, 3\}$, which form the partition $\{\{1\}, \{2, 3\}\}$.

We get an equivalence relation by defining $x \sim y$ if for all $E \in \mathcal{B}$ $x \in E$ if and only if $y \in E$.

These three things are sort of the same (2/3)

In the example above we get $2 \sim 3$ but $1 \nsim 2$ and $1 \nsim 3$.
Given a partition $\mathcal{P}$ we can form an atomic algebra $\mathcal{B}$ whose elements are the sets $\bigcup_{E \in \mathcal{A}} E$ for some $\mathcal{A} \subseteq \mathcal{P}$. For example, the atomic algebra on $\{1, 2, 3\}$ corresponding to the partition $\{\{1\}, \{2, 3\}\}$ is $\{\varnothing, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$.
We can get an equivalence relation by defining $x \sim y$ if for all $E \in \mathcal{P}$ $x \in E$ if and only if $y \in E$. For example the equivalence relation for the partition above has $2 \sim 3$ but $1 \nsim 2$ and $1 \nsim 3$.
Given an equivalence relation we can get an atomic algebra by saying $E \in \mathcal{B}$ if $x \in E$ and $x \sim y$ imply $y \in E$. Starting from the relation $2 \sim 3$ but $1 \nsim 2$ and $1 \nsim 3$. we get the atomic algebra $\{\varnothing, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$.
We can get a partition by taking the set of all equivalence classes. For example, the equivalence classes for the relation above are $\{1\}$ and $\{2, 3\}$.

These three things are sort of the same (3/3)

As in the example, these constructions are all compatible in the sense that if I start, for example, with an atomic algebra, construct the corresponding equivalence relation, then the partition corresponding to that equivalence relation, and finally the atomic algebra corresponding to that partition, I get the same atomic algebra I started with. I could start at any other point and take any other path.

There is quite a lot to check here. It's all straightforward enough; there's just a lot of it. See the notes for the proofs.

More about atomic algebras

It's useful to have all three points of view available. Some things which look difficult from one point of view are obvious from another. The advantage of the atomic algebra point of view is its connection with Boolean algebras and σ-algebras.
Atomic algebras have similar properties to Boolean algebras or σ-algebras.
E.g. if $\mathcal{A} \subseteq \mathcal{B}$ then $\bigcap_{E \in \mathcal{A}} E \in \mathcal{B}$.
Also, the intersection of any collection of atomic algebras is an atomic algebra. For any $\mathcal{A} \in \wp(\wp(X))$ there's a smallest atomic algebra containing it. We say that this atomic algebra is *generated* by $\mathcal{A}$.

Natural order relations

There's a natural order structure on the atomic algebras on a set, i.e. $\mathcal{B}' \supseteq \mathcal{B}$.

There's a natural order structure on partitions, i.e. $\mathcal{Q}$ is a refinement of $\mathcal{P}$ if for every $E \in \mathcal{Q}$ there is an $F \in \mathcal{P}$ such that $E \subseteq F$.

There is a natural order structure on the set of equivalence relations, i.e. $\sim$ is stronger than $\bowtie$ if $x \sim y$ implies $x \bowtie y$.

These order relations are compatible with the constructions discussed earlier, e.g. If $\mathcal{B}_{\mathcal{P}}$ and $\mathcal{B}_{\mathcal{Q}}$ are the atomic algebras corresponding to partitions $\mathcal{P}$ and $\mathcal{Q}$ then $\mathcal{Q}$ is a refinement of $\mathcal{P}$ if and only if $\mathcal{B}_{\mathcal{Q}} \supseteq \mathcal{B}_{\mathcal{P}}$. Again, there is a lot to check, but it's mostly straightforward. See the notes for details.

Common refinement

Given two partitions $\mathcal{P}$ and $\mathcal{Q}$ there's a partition $\mathcal{R}$ whose elements are then non-empty intersections $E \cap F$ where $E \in \mathcal{P}$ and $F \in \mathcal{Q}$. As the same suggests, $\mathcal{R}$ is a refinement of both $\mathcal{P}$ and $\mathcal{Q}$.

The counterpart for atomic algebras is the algebra $\mathcal{B}_{\mathcal{R}}$ generated by $\mathcal{B}_{\mathcal{P}} \cup \mathcal{B}_{\mathcal{Q}}$.

The counterpart for equivalence relations is the weakest relation stronger than the equivalence relations corresponding to $\mathcal{P}$ and $\mathcal{Q}$, which exists by Proposition 2.3.2.

These operations show that atomic algebras, or partitions, or equivalence relations, form a directed set.

They would also be a directed set with the opposite order, but a less interesting one.

Systems of weights, measures

We defined Boolean algebras, closed under finite unions, and contents, which are defined on them and finitely additive. Then we defined σ-algebras, closed under countable unions, and measures, which are defined on them and countably additive. Now we've defined atomic algebras, closed under arbitrary unions, so should we define something measure-like which is defined on them and arbitrarily additive? If $w\colon X \to [0, +\infty]$ is a function then

$$\mu(E) = \sum_{x \in E} w(x)$$

would be an example of such an object. It's actually the only example! That's not what happened for contents and measures. We call a function $w\colon X \to [0, +\infty]$ a *system of weights* for X.