MAU22200 Lecture 45

John Stalker

Trinity College Dublin

15 February 2022

Completion (1/10)

Suppose $\mathcal{B}$ is a σ-algebra on a set X and μ is a measure on $(X, \mathcal{B})$. Let $\mathcal{B}^\dagger$ be the set of $F \in \wp(X)$ such that for every $\epsilon > 0$ there are $D, H \in \mathcal{B}$ such that $F \triangle H \subseteq D$ and $\mu(D) < \epsilon$. Then $\mathcal{B}^\dagger$ is a σ-algebra on X and $\mathcal{B} \subseteq \mathcal{B}^\dagger$. For $F \in \mathcal{B}^\dagger$ we define

$$\mu^-(F) = \sup_{\substack{E \in \mathcal{B} \\ E \subseteq F}} \mu(E), \qquad \mu^+(F) = \inf_{\substack{G \in \mathcal{B} \\ F \subseteq G}} \mu(G).$$

Then $\mu^-(F) = \mu^+(F)$ for all $F \in \mathcal{B}^\dagger$. Let $\mu^\dagger(F)$ be their common value. Then $\mu^\dagger$ is a measure on $(X, \mathcal{B}^\dagger)$ and $\mu^\dagger(F) = \mu(F)$ for all $F \in \mathcal{B}$.

$(X, \mathcal{B}^\dagger, \mu^\dagger)$ is called the *completion* of $(X, \mathcal{B}, \mu)$. We saw an almost identical theorem last week, but with Boolean algebras in place of σ-algebras and contents in place of measures.

Completion (2/10)

$\mathcal{B}$ is a Boolean algebra and μ is a content so last week's theorem shows that $\mathcal{B}^\dagger$ is a Boolean algebra on X, that $\mathcal{B} \subseteq \mathcal{B}^\dagger$, that $\mu^+(F) = \mu^-(F)$ for all $F \in \mathcal{B}^\dagger$, that $\mu^\dagger$ is a content on $(X, \mathcal{B}^\dagger)$ and that $\mu^\dagger(F) = \mu(F)$ for all $F \in \mathcal{B}$.
So we only need to show that $\mathcal{B}^\dagger$ is a σ-algebra rather than just a Boolean algebra and that $\mu^\dagger$ is a measure rather than just a content, i.e. that $\bigcup_{F \in \mathcal{A}} F \in \mathcal{B}^\dagger$ if $\mathcal{A}$ is a countable subset of $\mathcal{B}^\dagger$ and that $\mu^\dagger \left(\bigcup_{F \in \mathcal{A}} F\right) = \sum_{F \in \mathcal{A}} \mu^\dagger(F)$ if, in addition, the F's are disjoint. Only the countably infinite case is needed because for finite $\mathcal{A}$ we already have both statements, so we can take

$$\mathcal{A} = \{F_0, F_1, \ldots\}$$

for some sequence of distinct F's and prove that

$$\bigcup_{j=0}^{\infty} F_j \in \mathcal{B}^\dagger, \qquad \mu^\dagger\left(\bigcup_{j=0}^{\infty} F_j\right) = \sum_{j=0}^{\infty} \mu^\dagger(F_j).$$

Completion (3/10)

$F_i \in \mathcal{B}^\dagger$ so for any $\delta_i > 0$ there are $D_i, H_i \in \mathcal{B}$ such that $F_i \triangle H_i \subseteq D_i$ and $\mu(D_i) < \delta_i$. If $\epsilon > 0$ then $\delta_i = \frac{\epsilon}{2^{i+1}} > 0$ so we can choose D_i and H_i such that $F_i \triangle H_i \subseteq D_i$ and $\mu(D_i) < \frac{\epsilon}{2^{i+1}}$. Let

$$D = \bigcup_{i=0}^{\infty} D_i, \qquad F = \bigcup_{i=0}^{\infty} F_i, \qquad H = \bigcup_{i=0}^{\infty} H_i.$$

If $x \in F \triangle H$ then $x \in F$ and $x \notin H$ or $x \in H$ and $x \notin F$. If $x \in F$ and $x \notin H$ we have $x \in F_i$ for some i but $x \notin H_j$ for any j. In particular $x \notin H_i$ so $x \in F_i \triangle H_i$ and therefore $x \in D_i$ and $x \in D$. The same argument works if $x \in H$ and $x \notin F$, but with the roles of F and H reversed. So $F \triangle H \subseteq D$.

$$\mu(D) = \mu\left(\bigcup_{i=0}^{\infty} D_i\right) \leq \sum_{i=0}^{\infty} \mu(D_i) < \sum_{i=0}^{\infty} \frac{\epsilon}{2^{i+1}} = \epsilon.$$

So $F \in \mathcal{B}^\dagger$. Therefore $\mathcal{B}^\dagger$ is a σ-algebra.

Completion (4/10)

$\mu^\dagger(F) = \mu^-(F_i) \le \mu^-\left(\bigcup_{j=0}^\infty F_j\right) = \mu^\dagger\left(\bigcup_{j=0}^\infty F_j\right)$, since $F_i \subseteq \bigcup_{j=0}^\infty F_i$. If $\mu^\dagger(F_i) = +\infty$ for some i then $\mu^\dagger\left(\bigcup_{j=0}^\infty F_j\right) = +\infty$ and so $\mu^\dagger\left(\bigcup_{j=0}^\infty F_j\right) = \sum_{j=0}^\infty \mu^\dagger(F_j)$.

What's left is the case $\mu^\dagger(F_i) < +\infty$ for all i.

$\mu^+(F_i) = \inf_{\substack{G_i \in \mathcal{B} \\ F_i \subseteq G_i}} \mu(G_i)$. If $\epsilon > 0$ then $\mu^+(F_i) + \epsilon/2^{i+1}$ is greater than the infimum so there is a $G_i \in \mathcal{B}$ such that $F_i \subseteq G_i$ and $\mu(G_i) < \mu^+(F_i) + \frac{\epsilon}{2^{i+1}}$. Let $G = \bigcup_{i=0}^\infty G_i$. Then

$$\mu(G) \le \sum_{i=0}^\infty \mu(G_i) < \sum_{i=0}^\infty \left(\mu^+(F_i) + \frac{\epsilon}{2^{i+1}}\right) = \sum_{i=0}^\infty \mu^+(F_i) + \epsilon.$$

Now $F \subseteq G$ and $G \in \mathcal{B}$ so $\mu^+(F) \le \mu(G)$, and therefore

$$\mu^+(F) < \sum_{i=0}^\infty \mu^+(F_i) + \epsilon.$$

This holds for all $\epsilon > 0$ so $\mu^+(F) \le \sum_{i=0}^\infty \mu^+(F_i)$.

Completion (5/10)

Similarly, $\mu^-(F_i) = \sup_{\substack{E_i \in \mathcal{B} \\ E_i \subseteq F_i}} \mu(E_i)$. If $\epsilon > 0$ then $\mu^-(F_i) - \epsilon/2^{i+1}$ is less than the supremum so there is an $E_i \in \mathcal{B}$ such that $E_i \subseteq F_i$ and

$$\mu(E_i) < \mu^-(F_i) - \frac{\epsilon}{2^{i+1}}.$$

Let $E = \bigcup_{i=0}^{\infty} E_i$. Then

$$\mu(E) = \sum_{i=0}^{\infty} \mu(E_i) > \sum_{i=0}^{\infty} \left(\mu^-(F_i) - \frac{\epsilon}{2^{i+1}} \right) = \sum_{i=0}^{\infty} \mu^-(F_i) + \epsilon.$$

Here we've used the fact that the F's are disjoint so the E's, which are subsets of the F's, are also disjoint. Now $E \subseteq F$ and $F \in \mathcal{B}$ so $\mu^-(F) \geq \mu(E)$, and therefore

$$\mu^-(F) > \sum_{i=0}^{\infty} \mu^-(F_i) - \epsilon.$$

This holds for all $\epsilon > 0$ so $\mu^-(F) \geq \sum_{i=0}^{\infty} \mu^-(F_i)$.

Completion (6/10)

$F_i \in \mathcal{B}^\dagger$ for each i and $F \in \mathcal{B}^\dagger$, so

$$\mu^-(F_i) = \mu^\dagger(F_i) = \mu^+(F_i)$$
$$\mu^-(F) = \mu^\dagger(F) = \mu^+(F).$$

From

$$\mu^+(F) \leq \sum_{i=0}^{\infty} \mu^+(F_i)$$
$$\mu^-(F) \geq \sum_{i=0}^{\infty} \mu^-(F_i)$$

it therefore follows that

$$\mu^\dagger(F) = \sum_{i=0}^{\infty} \mu^\dagger(F_i).$$

Thus $\mu^\dagger$ is a measure on $(X, \mathcal{B}^\dagger)$.

Completion (7/10)

The statement of the theorem we just proved was chosen to look as much as possible like the one for Boolean algebras and contents. There are simpler versions if we only care about σ-algebras and measures.

> *Suppose that $(X, \mathcal{B}, \mu)$ and $(X, \mathcal{B}^\dagger, \mu^\dagger)$ are as in the preceding theorem. Then $F \in \mathcal{B}^\dagger$ if and only if there are $D, H \in \mathcal{B}$ such that $F \triangle H \subseteq D$ and $\mu(D) = 0$. Then $\mu^\dagger(F) = \mu(H)$.*

Suppose that there are $D, H \in \mathcal{B}$ such that $F \triangle H \subseteq D$ and $\mu(D) = 0$. For any $\epsilon > 0$ we have $\mu(D) < \epsilon$ so $F \in \mathcal{B}^\dagger$. Suppose, conversely, that $F \in \mathcal{B}^\dagger$. $1/2^{k+1} > 0$ so there are $D_k, H_k \in \mathcal{B}$ such that $F \triangle H_k \subseteq D_k$ and $\mu(D_k) < \frac{1}{2^{k+1}}$. Let $D = \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} D_j$ and $H = \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} H_j$. Note that $D, H \in \mathcal{B}$. $F \triangle H = \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} F \triangle H_i$. $F \triangle H_i \subseteq D_i$ so

$$\bigcup_{j=i}^{\infty} F \triangle H_i \subseteq \bigcup_{j=i}^{\infty} D_i, \qquad \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} F \triangle H_i \subseteq \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} D_i.$$

Completion (8/10)

$$\bigcap_{i=0}^{\infty}\bigcup_{j=i}^{\infty} F\triangle H_i \subseteq \bigcap_{i=0}^{\infty}\bigcup_{j=i}^{\infty} D_i, \qquad F\triangle H \subseteq D.$$

Also,

$$\mu(\bigcup_{j=i}^{\infty} D_j) \leq \sum_{j=i}^{\infty}\mu(D_j) \leq \sum_{j=i}^{\infty}\frac{1}{2^{j+1}} = \frac{1}{2^i}.$$

The sequence of sets $\bigcup_{j=i}^{\infty} D_j$ is monotone decreasing $\mu(D_0) < +\infty$ so

$$\mu(D) = \lim_{i\to\infty}\mu\left(\bigcup_{j=i}^{\infty} D_j\right) \leq \lim_{i\to\infty}\frac{1}{2^i} = 0.$$

So there are $D, H \in \mathcal{B}$ such that $F\triangle H \subseteq D$ and $\mu(D) = 0$. This completes the "if and only if" part of the statement. We still have to prove that $\mu^{\dagger}(F) = \mu(H)$.

Completion (9/10)

$\mu^\dagger(H \cup D) + \mu^\dagger(H \cap D) = \mu^\dagger(H) + \mu^\dagger(D)$ and $\mu^\dagger(D) = \mu(D) = 0$, from which it follows that $\mu^\dagger(H \cap D) = 0$ as well. Therefore $\mu^\dagger(H \cup D) = \mu^\dagger(H)$. Now $F \subseteq H \cup D$ so

$$\mu^\dagger(F) \leq \mu^\dagger(H \cup D) = \mu^\dagger(H) = \mu(H).$$

On the other hand,

$$\mu^\dagger(F \cup D) + \mu^\dagger(F \cap D) = \mu^\dagger(F) + \mu^\dagger(D),$$

$\mu^\dagger(D) = \mu(D) = 0$ and $\mu^\dagger(H \cap D) = 0$ so $\mu^\dagger(F \cup D) = \mu^\dagger(F)$. Now $H \subseteq F \cup D$ so

$$\mu(H) = \mu^\dagger(H) \leq \mu^\dagger(F \cup D) = \mu^\dagger(F).$$

Therefore $\mu^\dagger(F) = \mu(H)$. This completes the proof.

Completion (10/10)

Suppose that $(X, \mathcal{B}, \mu)$ and $(X, \mathcal{B}^\dagger, \mu^\dagger)$ are as in the preceding theorem. The following two statements are equivalent:

1. *$F \in \mathcal{B}^\dagger$ and $\mu^\dagger(F) = 0$.*
2. *There is a $G \in \mathcal{B}$ such that $F \subseteq G$ and $\mu(G) = 0$.*

Suppose $F \in \mathcal{B}^\dagger$ and $\mu^\dagger(F) = 0$. By the preceding proposition there are $D, H \in \mathcal{B}$ such that $F \triangle H \subseteq D$, $\mu(D) = 0$ and $\mu(H) = \mu(F) = 0$. Let $G = D \cup H$. Then $G \in \mathcal{B}$ and $\mu(G) = \mu(D \cup H) \leq \mu(D) + \mu(H) = 0$ and hence $\mu(G) = 0$. Also $F \subseteq G$.

Suppose conversely that there is a $G \in \mathcal{B}$ such that $F \subseteq G$ and $\mu(G) = 0$. Let $D = H = G$. Then $F \triangle H = G \setminus F \subseteq G = D$ and $\mu(D) = \mu(G) = 0$. So $F \in \mathcal{B}^\dagger$ by the preceding proposition. Also, $F \subseteq G$ so

$$\mu^\dagger(F) \leq \mu^\dagger(G) = \mu(G) = 0$$

and hence $\mu^\dagger(F) = 0$.