MAU22200 Lecture 44

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Properties of measures (1/7)

Suppose $(X, \mathcal{B}, \mu)$ is a measure space. Then

1. *If $E, F \in \mathcal{B}$ then* $\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$.
2. *If $E, F \in \mathcal{B}$ and $E \subseteq F$ then $\mu(E) \leq \mu(F)$.*
3. *If $E, F \in \mathcal{B}$ then $\mu(E \cup F) \leq \mu(E) + \mu(F)$.*
4. *If $\mathcal{A}$ is a countable subset of $\mathcal{B}$ and $E \cap F = \varnothing$ whenever $E, F \in \mathcal{A}$ and $E \cap F = \varnothing$ then* $\mu\left(\bigcup_{E \in \mathcal{A}} E\right) = \sum_{E \in \mathcal{A}} \mu(E)$.
5. *If $\mathcal{A}$ is a countable subset of $\mathcal{B}$ then* $\mu\left(\bigcup_{E \in \mathcal{A}} E\right) \leq \sum_{E \in \mathcal{A}} \mu(E)$.

The first three of these are properties of contents and all measures are contents, so we don't need to re-prove them. The fourth one is just part of the definition of a measure, so we don't need to prove it either.

Properties of measures (2/7)

Suppose $\mathcal{A}$ is a countable subset of $\mathcal{B}$. If $\mathcal{A}$ is finite then we've already proved $\mu\left(\bigcup_{E\in\mathcal{A}} E\right) \le \sum_{E\in\mathcal{A}} \mu(E)$ when we discussed contents. So we can assume $\mathcal{A}$ is countably infinite, i.e. $\mathcal{A} = \{E_1, E_2, \ldots\}$ for distinct $E_1, E_2, \in \mathcal{B}$. Set $G_j = E_j \setminus \bigcup_{i<j} E_i$. Then $G_j \subseteq E_j$. For every $x \in \bigcup_{j=1}^{\infty} E_j$ there is a first value of j for which $x \in E_j$ and so $x \in G_j$ for this j, so

$$\bigcup_{j=1}^{\infty} E_j \subseteq \bigcup_{j=1}^{\infty} G_j$$

$$\mu(G_j) \le \mu(E_j) \qquad \mu\left(\bigcup_{j=1}^{\infty} E_j\right) \le \mu\left(\bigcup_{j=1}^{\infty} G_j\right).$$

Now $G_j \cap G_k = \varnothing$ if $j \neq k$ so $\mu\left(\bigcup_{j=1}^{\infty} G_j\right) = \sum_{j=1}^{\infty} \mu(G_j)$. So

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \le \sum_{j=1}^{\infty} \mu(E_j).$$

Properties of measures (3/7)

Suppose $(X, \mathcal{B}, \mu)$ *is a measure space and* $E\colon \mathbf{N} \to \mathcal{B}$ *is a sequence of sets which is monotone increasing in the sense that* $E_j \subseteq E_k$ *if* $j \leq k$. *Then* $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$.

Again set $G_j = E_j \setminus \bigcup_{i<j} E_i$. As before, $\bigcup_{j=0}^{\infty} G_j = \bigcup_{j=0}^{\infty} E_j$ and the G's are disjoint so $\mu\left(\bigcup_{j=0}^{\infty} E_j\right) = \sum_{j=0}^{\infty} \mu(G_j)$. Similarly, $\bigcup_{j=0}^{m} G_j = \bigcup_{j=0}^{m} E_j$ and $\mu\left(\bigcup_{j=0}^{m} E_j\right) = \sum_{j=0}^{m} \mu(G_j)$. The monotonicity assumption on E means that $\bigcup_{j=0}^{m} E_j = E_m$, so

$$\mu\left(\bigcup_{j=0}^{\infty} E_j\right) = \sum_{j=0}^{\infty} \mu(G_j) = \lim_{m\to\infty} \sum_{j=0}^{m} \mu(G_j) = \lim_{m\to\infty} \mu(E_m).$$

Properties of measures (4/7)

Suppose $(X, \mathcal{B}, \mu)$ *is a measure space and* $E\colon \mathbf{N} \to \mathcal{B}$ *is a sequence of sets which is monotone decreasing in the sense that* $E_j \supseteq E_k$ *if* $j \leq k$*. If* $\mu(E_m) < +\infty$ *for some* m *then*

$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

Note that there's an additional hypothesis here compared to the preceding proposition, that $\mu(E_m) < +\infty$ for some m. This hypothesis is needed. There's a counterexample in the notes to to show that the theorem is not true without it. This proposition isn't as important as the one for increasing sequences, but it's useful practice in using the various properties.

Properties of measures (5/7)

Let $F_j = E_m \setminus E_{j+m}$. If $j \leq k$ then $j + m \leq k + m$ so $E_{j+m} \supseteq E_{k+m}$ and $F_j \subseteq F_k$. It follows from the proposition about increasing sequences that

$$\mu\left(\bigcup_{j=0}^{\infty} F_j\right) = \lim_{j\to\infty} \mu(F_j).$$

Now $E_m = F_j \cup E_{j+m}$ and $F_j \cap E_{j+m} = \varnothing$ so

$$\mu(E_m) = \mu(F_j) + \mu(E_{j+m}).$$

$$\mu(E_m) = \lim_{j\to\infty} \mu(F_j) + \lim_{j\to\infty} \mu(E_{j+m}) = \mu\left(\bigcup_{j=0}^{\infty} F_j\right) + \lim_{j\to\infty} \mu(E_j).$$

Properties of measures (6/7)

Now

$$\bigcup_{j=0}^{\infty} F_j = \bigcup_{j=0}^{\infty} (E_m \setminus E_{j+m}) = E_m \setminus \left(\bigcap_{j=0}^{\infty} E_{j+m} \right)$$

and $\bigcap_{j=0}^{\infty} E_{j+m} \subseteq E_m$ so

$$E_m = \left(\bigcap_{j=0}^{\infty} E_{j+m} \right) \cup \left(\bigcup_{j=0}^{\infty} F_j \right), \qquad \left(\bigcap_{j=0}^{\infty} E_{j+m} \right) \cap \left(\bigcup_{j=0}^{\infty} F_j \right) = \varnothing.$$

$$\mu(E_m) = \mu \left(\bigcap_{j=0}^{\infty} E_{j+m} \right) + \mu \left(\bigcup_{j=0}^{\infty} F_j \right).$$

If $x \in \bigcap_{j=0}^{\infty} E_{j+m}$ then $x \in E_m$ and, by the monotonicity assumption on E, $x \in E_k$ for all $k < m$. But also $x \in E_k$ for all $k \geq m$ because such k can be written as $j + m$ for some $m \in \mathbf{N}$. It follows that $x \in \bigcap_{k=0}^{\infty} E_k$. Therefore $\bigcap_{j=0}^{\infty} E_{j+m} \subseteq \bigcap_{k=0}^{\infty} E_k$.

Properties of measures (7/7)

We just proved $\bigcap_{j=0}^{\infty} E_{j+m} \subseteq \bigcap_{k=0}^{\infty} E_k$. The reverse inclusion holds as well because if $k = j + m$ and $j \geq 0$ then $k \geq 0$, so $\bigcap_{j=0}^{\infty} E_{j+m} = \bigcap_{k=0}^{\infty} E_k$. We therefore have

$$\mu(E_m) = \mu\left(\bigcap_{j=0}^{\infty} E_j\right) + \mu\left(\bigcup_{j=0}^{\infty} F_j\right),$$

We combine this with the equation

$$\mu(E_m) = \mu\left(\bigcup_{j=0}^{\infty} F_j\right) + \lim_{j\to\infty} \mu(E_j)$$

obtained earlier. Either of these equations, together with the fact that $\mu(E_m) < +\infty$, gives $\mu\left(\bigcup_{j=0}^{\infty} F_j\right) < +\infty$ so we can subtract it from both sides to obtain

$$\mu\left(\bigcap_{j=0}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

Some definitions

A measure space $(X, \mathcal{B}, \mu)$ is called finite if $\mu(X) < +\infty$ and is called σ-finite if there is a countable subset $\mathcal{A} \subseteq \mathcal{B}$ such that $X = \bigcup_{E \in \mathcal{A}} E$ and $\mu(E) < +\infty$ for all $E \in \mathcal{A}$.

Note that $(X, \mathcal{B}, \mu)$ does not mean that X is a finite set!

Suppose $(X, \mathcal{T})$ is a locally compact σ-compact Hausdorff topological space. Let $\mathcal{B}$ be the Borel σ-algebra on X. A measure μ on $(X, \mathcal{B})$ is called a Borel measure. If it also satisfies the following conditions then it is called a Radon measure:

1. *If K is a compact subset of X then $\mu(K) < +\infty$.*
2. *If $E \in \mathcal{B}$ then $\mu(E) = \sup \mu(K)$, where the supremum is over all compact subsets K of E.*
3. *If $E \in \mathcal{B}$ then $\mu(E) = \inf \mu(U)$, where the infimum is over all open supersets U of E.*

An important goal this semester is to construct Lebesgue measure. The construction will show that it is a Radon measure.