MAU22200 Lecture 43

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A natural content on $\mathcal{I}$ (1/2)

Last time we saw that the set $\mathcal{I}$ of finite unions of intervals in $\mathbf{R}$ is a Boolean algebra on $\mathbf{R}$. If $E \in \mathcal{I}$ then a partition of E is a finite set of disjoint intervals whose union is E. Every element of $\mathcal{I}$ has a such a partition, but it may have many such partitions, e.g.

$$\begin{aligned}[-1,1] &= [-1,0) \cup [0,1] = [-1,0] \cup (0,1] \\ &= [-1,-1/3] \cup (-1/3,1/3) \cup [1/3,1].\end{aligned}$$

There is a content μ on $(\mathbf{R}, \mathcal{I})$ such that if $\{I_1, \ldots, I_m\}$ is a partition of E then

$$\mu(E) = \sum_{k=1}^{m} \ell(I_k).$$

ℓ is the length function, defined in the obvious way, i.e. $\ell(I) = \sup I - \inf I$ for non-empty I and $\ell(\varnothing) = 0$.

A natural content on $\mathcal{I}$ (2/2)

$\mu(E) = \sum_{k=1}^{m} \ell(I_k)$ can't be used directly as a definition because there are multiple partitions of E and we don't know that the right hand side gives the same value for each possible partition. There are two possible ways around this:

- Show that if $\{I_1, \ldots, I_m\}$ and $\{J_1, \ldots, J_n\}$ are partitions of the same element of E then $\sum_{k=1}^{m} \ell(I_k) = \sum_{l=1}^{n} \ell(J_l)$, so there is no conflict.
- Define $\mu(E)$ directly in terms of E and show that if $\{I_1, \ldots, I_m\}$ is a partition of E then $\mu(E) = \sum_{k=1}^{m} \ell(I_k)$.

In the notes I chose the second option, but the first also works. I set $\mu(E)$ equal to the limit of 2^n the number of rationals in E where the denominator divides 2^n, and show that it is a content on $\mathcal{I}$ and that $\mu(E) = \sum_{k=1}^{m} \ell(I_k)$.

The Jordan algebra and Jordan content (1/2)

We proved the following theorem last time:

> *Suppose $\mathcal{B}$ is a Boolean algebra on a set X and μ is a content on $(X, \mathcal{B})$. Let $\mathcal{B}^\dagger$ be the set of $F \in \wp(X)$ such that for every $\epsilon > 0$ there are $D, H \in \mathcal{B}$ such that $F \triangle H \subseteq D$ and $\mu(D) < \epsilon$. Then $\mathcal{B}^\dagger$ is a Boolean algebra on X and $\mathcal{B} \subseteq \mathcal{B}^\dagger$. For $F \in \mathcal{B}^\dagger$ we define $\mu^-(F) = \sup_{E \in \mathcal{B}, E \subseteq F} \mu(E)$ and $\mu^+(F) = \inf_{G \in \mathcal{B}, F \subseteq G} \mu(G)$. Then $\mu^-(F) = \mu^+(F)$ for all $F \in \mathcal{B}^\dagger$. Let $\mu^\dagger(F)$ be their common value. Then $\mu^\dagger$ is a content on $(X, \mathcal{B})$ and $\mu^\dagger(F) = \mu(F)$ for all $F \in \mathcal{B}$.*

We can apply this to $(X, \mathcal{B}, \mu) = (\mathbf{R}, \mathcal{I}, \mu)$. The resulting $\mathcal{B}^\dagger$ is called the *Jordan algebra* on $\mathbf{R}$, denoted $\mathcal{J}$, and the resulting $\mu^\dagger$ is called the *Jordan content*.

The Jordan algebra and Jordan content (2/2)

The Jordan algebra $\mathcal{J}$ is strictly larger than the algebra $\mathcal{I}$ of finite unions of intervals. The Cantor set C belongs to $\mathcal{J}$ but not to $\mathcal{I}$. Its Jordan content is 0.

To see this, we note that for every $\epsilon > 0$ there is an $n \in \mathbf{N}$ such that $(2/3)^n < \epsilon$. We can then take $D = H = C_n$ and check that $D, H \in \mathcal{I}$, $C \triangle H \subseteq D$, and $\mu(D) = (2/3)^n < \epsilon$. Therefore $C \in \mathcal{J}$. Also, $\mu^\dagger(C) = \inf_{G \in \mathcal{I}, C \subseteq G} \mu(G) \leq (2/3)^n$ for all n.

Not every subset of $\mathbf{R}$ belongs to $\mathcal{J}$. Many interesting subsets do not. E.g. $\mathbf{Q} \notin \mathcal{J}$. The proof is given in the notes.

You should think of $\mathcal{J}$ as the object of interest and $\mathcal{I}$ merely as scaffolding used in the construction of $\mathcal{J}$.

Banach-Tarski

Theorem (Banach-Tarski): There are subsets E_1, E_2, E_3, E_4, E_5 and F_1, F_2, F_3, F_4, F_5 of $\mathbf{R}^3$ such that

- E_i is congruent to F_i for each i,
- $E_i \cap E_j = \varnothing$ and $F_i \cap F_j = \varnothing$ when $i \neq j$,
- $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ is a ball of radius 1.
- $F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5$ is the union of two balls of radius 1, which do not intersect.

Suppose $\mathcal{B}$ is a Boolean algebra on $\mathbf{R}^3$, μ is a content, and

- congruent sets have the same content, and
- balls have a content equal to their volume.

If $E_1, \ldots, E_5 \in \mathcal{B}$ then we can prove that

$$\frac{4}{3}\pi = \frac{8}{3}\pi.$$

Equivalently, we know that $\frac{4}{3}\pi \neq \frac{8}{3}\pi$ so at least one E_i is not in $\mathcal{B}$. Any reasonable notion of volume is a content with the two properties above, so can't be defined for all subsets of $\mathbf{R}^3$.

Measures (1/4)

Contents are defined on Boolean-algebras. The corresponding notion for σ-algebras is a *measure*.

μ is a measure on $(X, \mathcal{B})$ if

- $\mu(\varnothing) = 0$, and
- if $\mathcal{A} \subseteq \mathcal{B}$ is a countable collection of disjoint sets then $\mu\left(\bigcup_{E \in \mathcal{A}} E\right) = \sum_{E \in \mathcal{A}} \mu(E)$.

This is a generalisation because every measure on a σ-algebra is a content, but not every content is a measure. If you take $\mathcal{A} = \{E, F\}$ where $E, F \in \mathcal{B}$ and $E \cap F = \varnothing$ you get $\mu(E \cup F) = \mu(E) + \mu(F)$ from the definition of a content.

Measures (2/4)

Here's a content which is not a measure: Take $X = \mathbf{N}$ and $\mathcal{B} = \wp(\mathbf{N})$. $\mathcal{B}$ is a σ-algebra. Define

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is finite,} \\ +\infty & \text{if } E \text{ is infinite.} \end{cases}$$

$\mu(\varnothing) = 0$. Suppose $E, F \in \wp(\mathbf{N})$. If E and F are finite then $E \cup F$ is finite and $\mu(E \cup F) = 0 = 0 + 0 = \mu(E) + \mu(F)$. If E is finite and F is infinite then $E \cup F$ is infinite and $\mu(E \cup F) = +\infty = 0 + +\infty = \mu(E) + \mu(F)$. If E is infinite and F is finite then $E \cup F$ is infinite and $\mu(E \cup F) = +\infty = +\infty + 0 = \mu(E) + \mu(F)$. If E and F are infinite then $E \cup F$ is infinite and $\mu(E \cup F) = +\infty = +\infty + +\infty = \mu(E) + \mu(F)$. So $\mu(E \cup F) = \mu(E) + \mu(F)$ in all cases and μ is a content.

Measures (3/4)

Let $\mathcal{A}$ be the set of singletons in $\mathbf{N}$, i.e. sets with only one element. They're all disjoint, i.e. if $E, F \in \mathcal{A}$ and $E \neq F$ then $E \cap F = \varnothing$. $\bigcup_{E \in \mathcal{A}} = \mathbf{N}$, which is infinite, so $\mu\left(\bigcup_{E \in \mathcal{A}}\right) = +\infty$. But $\sum_{E \in \mathcal{A}} \mu(E) = \sum_{E \in \mathcal{A}} 0 = 0$, since every element of $\mathcal{A}$ is finite. $\mu\left(\bigcup_{E \in \mathcal{A}}\right) \neq \sum_{E \in \mathcal{A}} \mu(E)$ so μ is not a measure.
If $(X, \mathcal{B}, \mu)$ is a measure space then $E \in \mathcal{B}$ is called a *null set* if $\mu(E) = 0$. The empty set is always a null set. The union of countably many disjoint null sets is a null set. We'll see soon that we can drop the word "disjoint".
Suppose $\mathcal{B}$ is σ-algebra on X and $x \in X$. Suppose $w \colon X \to [0, +\infty]$ is a function. The following are measures on $(X, \mathcal{B})$:

- $\mu(E) = 0$ for all $E \in \mathcal{B}$,
- $\mu(\varnothing) = 0$ and $\mu(E) = +\infty$ for all other $E \in \mathcal{B}$,
- $\mu(E) = 1$ if a $x \in E$ and $\mu(E) = 0$ if $x \notin E$.

Measures (4/4)

- $\mu(E) = n$ if E is a finite set in $\mathcal{B}$ with n elements and $\mu(E) = +\infty$ if E is an infinite set in $\mathcal{B}$.
- $$\mu(E) = \sum_{x \in E} w(x).$$

We saw in Lecture 41 that they were contents, but they're also measures. The first and second examples are not very interesting but the third example is known as *Dirac measure* and the fourth are known as *counting measure*. The first four examples are all special cases of the fifth.

The null sets in these five examples are, in order, every subset, only the empty set, the sets which don't contain x, only the empty set, and the subsets of $w^*(\{0\})$.

The measure we're most interested in, Lebesgue measure, won't appear until after Reading Week.