MAU22200 Lecture 42

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8 February 2022

Properties of contents (1/2)

Suppose X is a set, $\mathcal{B}$ is a Boolean algebra on X and μ is a content on $(X, \mathcal{B})$.

- *Monotonicity:* If $E, F \in \mathcal{B}$ and $E \subseteq F$ then $\mu(E) \leq \mu(F)$.
  $F = E \cup (F \setminus E)$ and $E \cap (F \setminus E) = \varnothing$ so
  $\mu(F) = \mu(E) + \mu(F \setminus E)$. $\mu(F \setminus E) \geq 0$ so $\mu(E) \leq \mu(F)$.
- *Finite additivity:* If $E_1, E_2, \ldots, E_m$ are disjoint elements of $\mathcal{B}$ then $\mu\left(\bigcup_{i=1}^m E_i\right) = \sum_{i=1}^m \mu(E_i)$.
  This is true when $m = 0$ and the general case is proved by induction on m.
- *Finite subadditivity:* If $E_1, E_2, \ldots, E_m$ are elements of $\mathcal{B}$ then $\mu\left(\bigcup_{i=1}^m E_i\right) \leq \sum_{i=1}^m \mu(E_i)$.
  Let $G_i = E_i \setminus \left(\bigcup_{j<i} E_j\right)$. Then the G's are disjoint elements of $\mathcal{B}$ and $\bigcup_{i=1}^m G_i = \bigcup_{i=1}^m E_i$ so $\mu\left(\bigcup_{i=1}^m E_i\right) = \sum_{i=1}^m \mu(G_i)$.
  But $G_i \subseteq E_i$ so $\mu(G_i) \leq \mu(E_i)$ and
  $\sum_{i=1}^m \mu(G_i) \leq \sum_{i=1}^m \mu(E_i)$.

Properties of contents (2/2)

- *The union/intersection property:* If $E, F \in \mathcal{B}$ then $\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$.
  From $E \cup F = E \cup (F \setminus E)$ and $E \cap (F \setminus E) = \varnothing$ it follows that
  $$\mu(E \cup F) = \mu(E) + \mu(F \setminus E).$$
  From $F = (F \setminus E) \cup (E \cap F)$ and $(F \setminus E) \cup (E \cap F) = \varnothing$ it follows that
  $$\mu(F) = \mu(F \setminus E) + \mu(E \cap F).$$
  So
  $$\mu(E)+\mu(F) = \mu(E)+\mu(F\setminus E)+\mu(E\cap F) = \mu(E\cup F)+\mu(E\cap F).$$

Completion (1/7)

Suppose $\mathcal{B}$ is a Boolean algebra on a set X and μ is a content on $(X, \mathcal{B})$. Let $\mathcal{B}^\dagger$ be the set of $F \in \wp(X)$ such that for every $\epsilon > 0$ there are $D, H \in \mathcal{B}$ such that $F \triangle H \subseteq D$ and $\mu(D) < \epsilon$. Then $\mathcal{B}^\dagger$ is a Boolean algebra on X and $\mathcal{B} \subseteq \mathcal{B}^\dagger$. For $F \in \mathcal{B}^\dagger$ we define

$$\mu^-(F) = \sup_{\substack{E \in \mathcal{B} \\ E \subseteq F}} \mu(E)$$

$$\mu^+(F) = \inf_{\substack{G \in \mathcal{B} \\ F \subseteq G}} \mu(G).$$

Then $\mu^-(F) = \mu^+(F)$ for all $F \in \mathcal{B}^\dagger$. Let $\mu^\dagger(F)$ be their common value. Then $\mu^\dagger$ is a content on $(X, \mathcal{B})$ and

$$\mu^\dagger(F) = \mu(F)$$

for all $F \in \mathcal{B}$.

$(X, \mathcal{B}^\dagger, \mu^\dagger)$ is called the *completion* of $(X, \mathcal{B}, \mu)$.

Completion (2/7)

There's a long list of things to show:

- $\mathcal{B} \subseteq \mathcal{B}^\dagger$
- $\mathcal{B}^\dagger$ is a Boolean algebra on X.
- If $F \in \mathcal{B}^\dagger$ then $\mu^-(F) = \mu^+(F)$.
- $\mu^\dagger$ is a content on $(X, \mathcal{B}^\dagger)$.
- If $F \in \mathcal{B}$ then $\mu^\dagger(F) = \mu(F)$.

First we show that $\mathcal{B} \subseteq \mathcal{B}^\dagger$. For any $F \in \mathcal{B}$ and $\epsilon > 0$ we choose $D = \varnothing$ and $H = F$. Then $D, H \in \mathcal{B}$,

$$F \triangle H = \varnothing \subseteq D$$

and

$$\mu(D) = \mu(\varnothing) = 0 < \epsilon$$

so $F \in \mathcal{B}^\dagger$. So if $F \in \mathcal{B}$ then $F \in \mathcal{B}^\dagger$. In other words, $\mathcal{B} \subseteq \mathcal{B}^\dagger$.

Completion (3/7)

Next we show that $\mathcal{B}^\dagger$ is a Boolean algebra on X. $\varnothing \in \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{B}^\dagger$ so $\varnothing \in \mathcal{B}^\dagger$.

Suppose $F \in \mathcal{B}^\dagger$, i.e. that for all $\epsilon > 0$ there are $D, H \in \mathcal{B}$ such that $F \triangle H \subseteq D$ and $\mu(D) < \epsilon$. Then

$$(X \setminus F) \triangle (X \setminus H) = F \triangle H \subseteq D,$$

$X \setminus H \in \mathcal{B}$ and $\mu(D) < \epsilon$. So $X \setminus F \in \mathcal{B}^\dagger$.

Suppose $F_1, F_2 \in \mathcal{B}^\dagger$, i.e. that for any $\delta > 0$ there are $D_1, H_1, D_2, H_2 \in \mathcal{B}$ such that $F_i \triangle H_i \subseteq D_i$ and $\mu(D_i) < \delta$. If $\epsilon > 0$ then $\epsilon/2 > 0$ so there are $D_1, H_1, D_2, H_2 \in \mathcal{B}$ such that $F_i \triangle H_i \subseteq D_i$ and $\mu(D_i) < \epsilon/2$. Let $D = D_1 \cup D_2$ and $H = H_1 \cup H_2$. Then $D, H \in \mathcal{B}$,

$$\begin{aligned}(F_1 \cup F_2) \triangle H = (F_1 \cup F_2) \triangle (H_1 \cup H_2) &\subseteq (F_1 \triangle H_1) \cup (F_2 \triangle H_2) \\ &\subseteq D_1 \cup D_2 = D\end{aligned}$$

$$\mu(D) = \mu(D_1 \cup D_2) \leq \mu(D_1) + \mu(D_2) < \epsilon/2 + \epsilon/2 = \epsilon.$$

So for every $\epsilon > 0$ there are $D, H \in \mathcal{B}$ such that $(F_1 \cup F_2) \triangle H \subseteq D$ and $\mu(D) < \epsilon$. Therefore $F_1 \cup F_2 \in \mathcal{B}^\dagger$.

Completion (4/7)

Next we show that if $F \in \mathcal{B}^\dagger$ then $\mu^-(F) = \mu^+(F)$. If $E, G \in \mathcal{B}$ and $E \subseteq F \subseteq G$ then $E \subseteq G$ and hence $\mu(E) \subseteq \mu(G)$. Taking the supremum over E and the infimum over G gives

$$\mu^-(F) = \sup_{\substack{E \in \mathcal{B} \\ E \subseteq F}} \mu(E) \leq \inf_{\substack{G \in \mathcal{B} \\ F \subseteq G}} \mu(G) = \mu^+(F).$$

So $\mu^-(F) \leq \mu^+(F)$.
By hypothesis $F \in \mathcal{B}^\dagger$ so for any $\epsilon > 0$ there are $D, H \in \mathcal{B}$ such that $F \triangle H \subseteq D$ and $\mu(D) < \epsilon$. Let $E = H \setminus D$ and $G = H \cup D$. Then $E, G \in \mathcal{B}$ and $E \subseteq F \subseteq G$ so $\mu(E) \leq \mu^-(F)$ and $\mu^+(F) \leq \mu(G)$. On the other hand, $G = E \cup D$ so

$$\mu(G) \leq \mu(E) + \mu(D) < \mu(E) + \epsilon.$$

Combining these inequalities,

$$\mu^+(F) < \mu^-(F) + \epsilon.$$

This holds for all $\epsilon > 0$ so $\mu^+(F) \leq \mu^-(F)$. Together with the reverse inequality, proved above, this gives $\mu^+(F) = \mu^-(F)$.

Completion (5/7)

Next we show that $\mu^\dagger$ is a content.

$$\mu^\dagger(\varnothing) = \mu^-(\varnothing) = \sup_{\substack{E\in\mathcal{B}\\ E\subseteq\varnothing}} \mu(E) = \sup_{E=\varnothing} \mu(E) = \mu(\varnothing) = 0.$$

Suppose $F_1, F_2 \in \mathcal{B}^\dagger$ and $F_1 \cap F_2 = \varnothing$.

$$\{E \in \mathcal{B} \colon E \subseteq F_i\} \subseteq \{E \in \mathcal{B} \colon E \subseteq F_1 \cup F_2\}$$

so

$$\mu^\dagger(F_i) = \mu^-(F_i) = \sup_{\substack{E\in\mathcal{B}\\ E\subseteq F_i}} \mu(E) \leq \sup_{\substack{E\in\mathcal{B}\\ E\subseteq F_1\cup F_2}} \mu(E) = \mu^-(F_1\cup F_2) = \mu^\dagger(F_1\cup F_2)$$

It follows that if $\mu^\dagger(F_i) = +\infty$ then $\mu^\dagger(F_1 \cup F_2) = +\infty$. So

$$\mu^\dagger(F_1 \cup F_2) = \mu^\dagger(F_1) + \mu^\dagger(F_2).$$

Completion (6/7)

The proof that

$$\mu^\dagger(F_1 \cup F_2) = \mu^\dagger(F_1) + \mu^\dagger(F_2)$$

when $\mu^\dagger(F_1)$ and $\mu^\dagger(F_2)$ are less than $+\infty$ is more complicated. The crucial idea is to prove

$$\mu^-(F_1 \cup F_2) \geq \mu^-(F_1) + \mu^-(F_2)$$

and

$$\mu^+(F_1 \cup F_2) \leq \mu^+(F_1) + \mu^+(F_2).$$

separately. $\mu^\dagger = \mu^+ = \mu^-$ so it then follows that

$$\mu^\dagger(F_1 \cup F_2) = \mu^\dagger(F_1) + \mu^\dagger(F_2).$$

See the notes for the proof of the two inequalities above.

Completion (7/7)

Finally we show that $\mu^\dagger(F) = \mu(F)$ if $F \in \mathcal{B}$. If $E \in \mathcal{B}$ and $E \subseteq F$ then $\mu(E) \leq \mu^-(F)$, by the definition of μ^-. Taking $E = F$ gives $\mu(F) \leq \mu^-(F)$. Similarly, if $G \in \mathcal{B}$ and $F \subseteq G$ then $\mu^+(F) \leq \mu(G)$. Taking $G = F$ gives $\mu^+(F) \leq \mu(F)$. But

$$\mu^-(F) = \mu^\dagger(F) = \mu^+(F)$$

so

$$\mu^\dagger(F) = \mu(F).$$