MAU22200 Lecture 41

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Constructions with Boolean algebras

The intersection of any collection of Boolean algebras on a set X is again a Boolean algebra on X.

The union of a collection of Boolean algebras needn't be a Boolean algebra.

$$\mathcal{B}_1 = \{\varnothing, \{1\}, \{2,3\}, \{1,2,3\}\}$$
$$\mathcal{B}_2 = \{\varnothing, \{2\}, \{1,3\}, \{1,2,3\}\}$$

are Boolean algebras on $\{1,2,3\}$, but $\mathcal{B}_1 \cup \mathcal{B}_2$ is not a Boolean algebra. $\{1\} \in \mathcal{B}_1 \cup \mathcal{B}_2$ and $\{2\} \in \mathcal{B}_1 \cup \mathcal{B}_2$ but $\{1,2\} \notin \mathcal{B}_1 \cup \mathcal{B}_2$.

Note that $\mathcal{B}_1 \cap \mathcal{B}_2 = \{\varnothing, \{1,2,3\}\}$ is a Boolean algebra, as it must be.

For any set of subsets of X there is a smallest Boolean algebra which contains them. This set is said to *generate* the algebra.

The Boolean algebra $\mathcal{I}$ is generated by the set of intervals.

Boolean algebras and functions

We've already seen that f^{**} preserves a lot of structures. Boolean algebras are no exception.

> *Suppose X and Y are sets, $f : X \to Y$ is a function and $\mathcal{B}$ is a Boolean algebra on X. Then $f^{**}(\mathcal{B})$ is a Boolean algebra on Y.*

For a proof, see the notes.

σ-algebras (1/2)

$\mathcal{B} \in \wp(\wp(X))$ is called a *σ-algebra* if

- $\varnothing \in \mathcal{B}$,
- if $E \in \mathcal{B}$ then $X \setminus E \in \mathcal{B}$, and
- if $\mathcal{A}$ is a countable subset of $\mathcal{B}$ then $\bigcup_{E \in \mathcal{A}} E \in \mathcal{B}$.

This is the same as the definition of a Boolean algebra, except we allow countable unions rather than just the union of a pair of elements.

The prefix σ is often used to denote countable unions. We've seen this before with σ-compact sets and will see it again.

"Algebra" is more tenuous here than for Boolean algebras because operations on countably many operands don't normally play a role in Algebra.

Sets with two elements are countable, so every σ-algebra is a Boolean algebra. Not every Boolean algebra is a σ-algebra. For example, the Boolean algebra $\mathcal{I}$ of finite unions of intervals is not a σ-algebra.

σ-algebras (2/2)

σ-algebras are closed under countable intersections as well as countable unions:

> *Suppose $\mathcal{A}$ is a non-empty countable subset of $\mathcal{B}$. Then $\bigcap_{E\in\mathcal{A}} E \in \mathcal{B}$.*

This is true because $\bigcap_{E\in\mathcal{A}} E = X \setminus (\bigcup_{E\in\mathcal{A}}(X \setminus E))$.
Most of the properties of σ-algebras are analogous to those of Boolean algebras. For example,

- If X and Y are sets, $f\colon X \to Y$ is a function and $\mathcal{B}$ is a σ-algebra on X then $f^{**}(\mathcal{B})$ is a σ-algebra on Y.
- The intersection of any non-empty collection of σ-algebras on X is a σ-algebra on X.
- For any $\mathcal{A} \in \wp(\wp(A))$ there is a smallest σ-algebra containing $\mathcal{A}$. The set $\mathcal{A}$ is said to *generate* this σ-algebra.

The Borel σ-algebra (1/4)

Suppose $(X, \mathcal{T})$ is a topological space. The σ-algebra generated by $\mathcal{T}$ is called the *Borel σ-algebra*. Its elements are called *Borel sets*.

The Cantor set is a Borel set.

As we've already seen,

$$C = \bigcap_{n \in \mathrm{N}} C_n$$

where $C_0 = [0, 1]$, $C_1 = [0, 1/3] \cup [2/3, 1]$, $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, etc. Each open set is a Borel set, since the open sets generate the Borel sets. Each closed interval is a Borel set, since its complement is open. Every finite union of closed intervals is a Borel set, so C_n is a Borel set for each n. The intersection of countably many Borel sets is Borel, so C is Borel.
It's also possible, though difficult, to give examples of subsets of $\mathbf{R}$ which are *not* Borel.

The Borel σ-algebra (2/4)

Suppose $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$ *are sets and* $f : X \to Y$ *is a continuous function. If* E *is a Borel subset in* Y *then* $f^*(E)$ *is a Borel subset of* X*.*

Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be the Borel algebras on X and Y respectively. Let

$$\mathcal{A} = f^{**}(\mathcal{B}_X).$$

This is a σ-algebra on Y.

$$\mathcal{T}_X \subseteq \mathcal{B}_X$$

$$f^{**}(\mathcal{T}_X) \subseteq f^{**}(\mathcal{B}_X) = \mathcal{A}.$$

It was proved last term that if f is continuous then

$$\mathcal{T}_Y \subseteq f^{**}(\mathcal{T}_X).$$

The Borel σ-algebra (3/4)

From
$$f^{**}(\mathcal{T}_X) \subseteq \mathcal{A}$$
and
$$\mathcal{T}_Y \subseteq f^{**}(\mathcal{T}_X)$$
it follows that
$$\mathcal{T}_Y \subseteq \mathcal{A}.$$
$\mathcal{A}$ is a σ-algebra on Y and $\mathcal{B}_Y$ is the smallest σ-algebra on Y containing $\mathcal{T}_Y$ so
$$\mathcal{B}_Y \subseteq \mathcal{A} = f^{**}(\mathcal{B}_X).$$
If $E \in \mathcal{B}_Y$ then $E \in f^{**}(\mathcal{B}_X)$ so $f^*(E) \in \mathcal{B}_X$. In other words, if E is a Borel subset in Y then $f^*(E)$ is a Borel subset of X.

The Borel σ-algebra (4/4)

> *Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, E is a Borel subset of X and F is a Borel subset of Y. Then $E \times F$ is a Borel subset of $X \times Y$.*

To prove this, note that

$$E \times F = \pi_1^*(E) \cap \pi_2^*(F)$$

where π_1 and π_2 are the two projections of $E \times F$. They're continuous, so $\pi_1^*(E)$ and $\pi_2^*(F)$ are Borel subsets of $E \times F$ by the preceding proposition. So their intersection is also a Borel subset.

Content

If X is a set and $\mathcal{B}$ is a Boolean algebra on X then $\mu\colon \mathcal{B} \to [0, +\infty]$ is called a *content* on $(X, \mathcal{B})$ if

- $\mu(\varnothing) = 0$,
- if $E, F \in \mathcal{B}$ and $E \cap F = \varnothing$ then

$$\mu(E \cup F) = \mu(E) + \mu(F).$$

The first condition is almost redundant. Suppose there's at least one $E \in \mathcal{B}$ such that $\mu(E) < +\infty$. Then

$$\mu(E) + 0 = \mu(E) = \mu(E \cup \varnothing) = \mu(E) + \mu(\varnothing).$$

By Cancellation then $\mu(\varnothing) = 0$.
Contents are more often called *finitely additive measures*, but there are problems with this name.

Simple examples

Suppose $\mathcal{B}$ is Boolean algebra on X and $x \in X$. The following are contents on $(X, \mathcal{B})$:

- $\mu(E) = 0$ for all $E \in \mathcal{B}$,
- $\mu(\varnothing) = 0$ and $\mu(E) = +\infty$ for all other $E \in \mathcal{B}$,
- $\mu(E) = 1$ if a $x \in E$ and $\mu(E) = 0$ if $x \notin E$.
- $\mu(E) = n$ if E is a finite set in $\mathcal{B}$ with n elements and $\mu(E) = +\infty$ if E is an infinite set in $\mathcal{B}$.

Suppose $w \colon X \to [0, +\infty]$ is a function. Then

$$\mu(E) = \sum_{x \in E} w(x)$$

is a content. The last example on the list above is simply the special case where $w(x) = 1$ for all $x \in X$.