MAU22200 Lecture 40

John Stalker

Trinity College Dublin

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Tonelli's Theorem

Last time we saw that

> *Suppose $\mathcal{A}$ is a set of disjoint sets. In other words if $P, Q \in \mathcal{A}$ and $P \neq Q$ then $P \cap Q = \varnothing$. Let $S = \bigcup_{P \in \mathcal{A}} P$. Suppose $f : S \to [0, +\infty]$ is a function. Then*
>
> $$\sum_{s \in S} f(s) = \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s).$$

Tonelli's Theorem (for sums) says that

> *Suppose A and B are sets and $f : A \times B \to [0, +\infty]$ is a function. Then*
>
> $$\sum_{a \in A} \sum_{b \in B} f(a, b) = \sum_{(a,b) \in A \times B} f(a, b) = \sum_{b \in B} \sum_{a \in A} f(a, b).$$

To prove the first equation, we apply the theorem above with $S = A \times B$ and $\mathcal{A}$ is the set of subsets of S of the form $\{a\} \times B$ for $a \in A$. The second equation is similar.

Fubini's Theorem (1/4)

Fubini's Theorem (for sums) says that

Suppose A and B are sets and $g\colon A \times B \to \mathbf{R}$ is a function such that

$$\sum_{(a,b)\in A\times B} g(a, b)$$

is convergent. Then

$$\sum_{a\in A} \sum_{b\in B} g(a, b) = \sum_{(a,b)\in A\times B} g(a, b) = \sum_{b\in B} \sum_{a\in A} g(a, b).$$

Convergent sums are absolutely convergent, so

$$\sum_{(a,b)\in A\times B} |g(a, b)| < +\infty.$$

Fubini's Theorem (2/4)

Let $S = A \times B$ and D the set of finite subsets of S, ordered by inclusion. Define $f : D \times S \to \mathbf{R}$ by

$$f(H, (a, b)) = \begin{cases} g(a, b) & \text{if } (a, b) \in H, \\ 0 & \text{if } (a, b) \notin H. \end{cases}$$

Then

$$\lim_{H \in D} f(H, (a, b)) = g(a, b)$$

and

$$|f(H, (a, b))| \leq |g(a, b)|.$$

We can apply the Dominated Convergence Theorem to get

$$\lim_{H \in D} \sum_{(a,b) \in S} f(H, (a, b)) = \sum_{(a,b) \in S} \lim_{H \in D} f(H, (a, b)).$$

Note that this is the version of the DCT for nets, not sequences, i.e. the one proved in the notes, not the one from lecture.

Fubini's Theorem (3/4)

Similarly, using the Dominated Convergence Theorem twice gives

$$\lim_{H\in D}\sum_{a\in A}\sum_{b\in B} f(H,(a,b)) = \sum_{a\in A}\lim_{H\in D}\sum_{b\in B} f(H,(a,b)) = \sum_{a\in A}\sum_{b\in B}\lim_{H\in D} f(H,(a,b)).$$

$$\lim_{H\in D}\sum_{b\in B}\sum_{a\in A} f(H,(a,b)) = \sum_{b\in B}\sum_{a\in A}\lim_{H\in D} f(H,(a,b)).$$

$$\sum_{a\in A}\sum_{b\in B} f(H,(a,b)) = \sum_{(a,b)\in A\times B} f(H,(a,b)) = \sum_{b\in B}\sum_{a\in A} f(H,(a,b))$$

because these are *finite* sums, so the order of summation doesn't matter. We take the limit over $H \in D$.

Fubini's Theorem (4/4)

$$\sum_{a\in A}\sum_{b\in B}\lim_{H\in D} f(H,(a,b)) = \sum_{(a,b)\in A\times B}\lim_{H\in D} f(H,(a,b))$$
$$= \sum_{b\in B}\sum_{a\in A}\lim_{H\in D} f(H,(a,b))$$

But $\lim_{H\in D} f(H,(a,b)) = g(a,b)$, so

$$\sum_{a\in A}\sum_{b\in B} g(a,b) = \sum_{(a,b)\in A\times B} g(a,b) = \sum_{b\in B}\sum_{a\in A} g(a,b).$$

So we're done.

Boolean algebras

A *Boolean algebra* on a set X is an element $\mathcal{B}$ of $\wp(\wp(X))$, i.e. a set of subsets of X, such that

- $\varnothing \in \mathcal{B}$,
- If $E \in \mathcal{B}$ then $X \setminus E \in \mathcal{B}$, and
- If $E, F \in \mathcal{B}$ then $E \cup F \in \mathcal{B}$.

This definition is minimal, to make it easier to prove something is a Boolean algebra. For example, it omits the fact that if $E, F \in \mathcal{B}$ then $E \cap F \in \mathcal{B}$, because that follows from the conditions above.

$$E \cap F = X \setminus ((X \setminus E) \cup (X \setminus F)).$$

By induction, finite unions and intersections of elements of $\mathcal{B}$ belong to $\mathcal{B}$. Similarly $E \setminus F \in \mathcal{B}$ and $E \triangle F \in \mathcal{B}$. Also $X \in \mathcal{B}$.

Why "Boolean"?

We can relate logical operations to set operations, by identifying statements with the subset of elements for which they are true. This was discussed in Lecture 10.
AND, OR and NOT correspond to union, intersection and (relative) complement. These are called Boolean operators, because of their connection with George Boole's *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities*. You've probably used them in search engines of various kinds.
A Boolean algebra is (roughly) one whose set membership statements are closed under Boolean operators. Don't take this too seriously! The set of valid statements in a language with a finite alphabet must be countable, but Boolean algebras can be uncountable, e.g. $\wp(\mathbf{N})$.

Why "algebra"?

What do Boolean algebras have to do with algebras? The operations $\triangle$ and $\cap$ satisfy the relations

- $(A \triangle B) \triangle C = A \triangle (B \triangle C)$,
- $A \triangle B = B \triangle A$,
- $A \triangle \varnothing = A$,
- $A \triangle A = \varnothing$,
- $(A \cap B) \cap C = A \cap (B \cap C)$,
- $A \cap B = B \cap A$,
- $X \cap A = A$,
- $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

In other words, a Boolean algebra $\mathcal{B}$ on X is a commutative ring, with $\triangle$ as addition, $\cap$ as multiplication, $\varnothing$ as additive identity and X as multiplicative identity. $\mathcal{F} = \{\varnothing, X\}$ is a subring of $\mathcal{B}$, and is a field. $\mathcal{B}$ is an algebra over $\mathcal{F}$.

Intervals (1/2)

The set of intervals in $\mathbf{R}$ is not a Boolean algebra. It's not closed under complements or unions, since the complement of an interval or the union of two intervals needn't be an interval. The set $\mathcal{I}$ of finite unions of intervals is a Boolean algebra though. This is painful to prove by brute force since there are just too many types of interval.

The simplest way to prove it is to find a single property which characterises intervals. There are a few options: $I \in \wp(\mathbf{R})$ is an interval if and only if

- I is connected
- I is convex
- I is such that if $x \leq y \leq z$ and $x, z \in I$ then $y \in I$

The third option is the easiest to work with. This would apply equally well with $[-\infty, +\infty]$ in place of $\mathbf{R}$.

Intervals (2/2)

See the notes for

- A proof that I satisfies the third condition if and only if it is of one of the ten forms (a, b), $[a, b]$, $[a, b)$, $(a, b]$, $(a, +\infty)$, $[a, +\infty)$, $(-\infty, b)$, $(-\infty, b]$, $(-\infty, +\infty)$ or $\varnothing$.
- A proof of the geometrically plausible fact that the complement of an interval is the union of two (possibly empty) intervals.
- A proof that the intersection of arbitrarily many intervals is an interval.

We don't need the last of these facts in showing that $\mathcal{I}$. is a Boolean algebra.

Consider the language obtained by taking inequalities, strict or weak, between a variable x and a constant, and connecting them by Boolean operators. $E \in \mathcal{I}$ if and only if E is the set of $x \in \mathbf{R}$ satisfying a statement in this language.

A weird fact

If $E \in \wp(\mathbf{R})$ is a union of m intervals then $X \setminus E$ is a union $m+1$ intervals. Some of them might be empty. In the notes I prove that it's a union of finitely many intervals but I don't prove the optimal bound of $m+1$.
It seems obvious then that if E is a union of countably many intervals then $X \setminus E$ must be a union of countably many intervals. But this is not true! If C is the Cantor set and $E = \mathbf{R} \setminus C$ then E is the union of countably many intervals but $X \setminus C$ is not!