MAU22200 Lecture 39

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The Dominated Convergence Theorem (1/6)

Suppose S *is a set and* $f\colon \mathbf{N} \times S \to \mathbf{R}$ *and* $g\colon S \to \mathbf{R}$ *are functions such that* $\lim_{n\to\infty} f_n(s)$ *exists for all* $s \in S$, $\sum_{s\in S} g(s) < +\infty$ *and*

$$|f_n(s)| \leq g(s)$$

for all $n \in \mathbf{N}$ *and* $s \in S$. *Then*

$$\lim_{n\to\infty} \sum_{s\in S} f_n(s) = \sum_{s\in S} \lim_{n\to\infty} f_n(s).$$

This is known as the Dominated Convergence Theorem (for sums). Note that our counterexample from Lecture 37 doesn't satisfy the hypotheses of this theorem. It was $S = \mathbf{N}$ and

$$f_n(s) = \begin{cases} 1 & \text{if } s = n, \\ 0 & \text{if } s \neq n. \end{cases}$$

The Dominated Convergence Theorem (2/6)

$$f_n(s) = \begin{cases} 1 & \text{if } s = n, \\ 0 & \text{if } s \neq n. \end{cases}$$

If $|f_n(s)| \leq g(s)$ for all $n \in \mathbf{N}$ and $s \in S$ then $g(s) \geq 1$ for all s.
But then $\sum_{s \in S} g(s) = +\infty$.
This example also didn't satisfy the hypotheses of the Monotone Convergence Theorem. That's a good thing, not a bad thing. It doesn't satisfy the conclusion of either theorem, so if it satisfied the hypotheses then we'd have a contradiction.
It does satisfy the hypotheses of Fatou's Lemma, but that's okay because Fatou's Lemma doesn't have

$$\lim_{n \to \infty} \sum_{s \in S} f_n(s) = \sum_{s \in S} \lim_{n \to \infty} f_n(s)$$

in its conclusion.

The Dominated Convergence Theorem (3/6)

Define

$$h_n(s) = g(s) + f_n(s)$$

$h_n(s) \geq 0$ so we can apply Fatou's Lemma:

$$\sum_{s \in S} \sup_{m \in \mathbb{N}} \inf_{n \geq m} h_n(s) \leq \sup_{m \in \mathbb{N}} \inf_{n \geq m} \sum_{s \in S} h_n(s).$$

$$\sup_{m \in \mathbb{N}} \inf_{n \geq m} h_n(s) = g(s) + \sup_{m \in \mathbb{N}} \inf_{n \geq m} f_n(s)$$

$$\sum_{s \in S} h_n(s) = \sum_{s \in S} g(s) + \sum_{s \in S} f_n(s).$$

$$\sum_{s \in S} \sup_{m \in \mathbb{N}} \inf_{n \geq m} h_n(s) = \sum_{s \in S} g(s) + \sum_{s \in S} \sup_{m \in \mathbb{N}} \inf_{n \geq m} f_n(s)$$

$$\sup_{m \in \mathbb{N}} \inf_{n \geq m} \sum_{s \in S} h_n(s) = \sum_{s \in S} g(s) + \sup_{m \in \mathbb{N}} \inf_{n \geq m} \sum_{s \in S} f_n(s).$$

The Dominated Convergence Theorem (4/6)

Combining the results from the previous slide,

$$\sum_{s \in S} g(s) + \sum_{s \in S} \sup_{m \in \mathrm{N}} \inf_{n \geq m} f_n(s) \leq \sum_{s \in S} g(s) + \sup_{m \in \mathrm{N}} \inf_{n \geq m} \sum_{s \in S} f_n(s).$$

The cancellation property isn't generally true in $[-\infty, +\infty]$, but we can cancel finite summands and $\sum_{s \in S} g(s)$ is finite.

$$\sum_{s \in S} \sup_{m \in \mathrm{N}} \inf_{n \geq m} f_n(s) \leq \sup_{m \in \mathrm{N}} \inf_{n \geq m} \sum_{s \in S} f_n(s).$$

Note that the conclusion is the same as for Fatou's Lemma, but f doesn't satisfy the hypothesis $f_n(s) \in [0, +\infty]$ from Fatou's Lemma, which is why we need the argument above. $\lim_{n \to \infty} f_n(s)$ exists so we can replace $\sup_{m \in \mathcal{N}} \inf_{n \geq m} f_n(s)$ with $\lim_{n \to \infty} f_n(s)$.

The Dominated Convergence Theorem (5/6)

We now have

$$\sum_{s \in S} \lim_{n \to \infty} f_n(s) \leq \sup_{m \in \mathbb{N}} \inf_{n \geq m} \sum_{s \in S} f_n(s).$$

The only properties of f we used are that $|f_n(s)| \leq g(s)$ for all $s \in S$ and that $\lim_{n \to \infty} f_n(s)$ exists. $-f$ has the same properties, so

$$\sum_{s \in S} \lim_{n \to \infty} -f_n(s) \leq \sup_{m \in \mathbb{N}} \inf_{n \geq m} \sum_{s \in S} -f_n(s).$$

$$-\sum_{s \in S} \lim_{n \to \infty} f_n(s) \leq -\inf_{m \in \mathbb{N}} \sup_{n \geq m} \sum_{s \in S} f_n(s).$$

$$\inf_{m \in \mathbb{N}} \sup_{n \geq m} \sum_{s \in S} f_n(s) \leq \sum_{s \in S} \lim_{n \to \infty} f_n(s).$$

The Dominated Convergence Theorem (6/6)

From

$$\sum_{s \in S} \lim_{n \to \infty} f_n(s) \leq \sup_{m \in \mathrm{N}} \inf_{n \geq m} \sum_{s \in S} f_n(s)$$

and

$$\inf_{m \in \mathrm{N}} \sup_{n \geq m} \sum_{s \in S} f_n(s) \leq \sum_{s \in S} \lim_{n \to \infty} f_n(s)$$

we get

$$\inf_{m \in \mathrm{N}} \sup_{n \geq m} \sum_{s \in S} f_n(s) \leq \sup_{m \in \mathrm{N}} \inf_{n \geq m} \sum_{s \in S} f_n(s)$$

The lemma from last time shows that

$$\lim_{n \to \infty} \sum_{s \in S} f_n(s)$$

exists and is equal to their common value. It follows that

$$\lim_{n \to \infty} \sum_{s \in S} f_n(s) = \sum_{s \in S} \lim_{n \to \infty} f_n(s)$$

Sums of sums (1/6)

Suppose $\mathcal{A}$ is a set of disjoint sets. In other words if $P, Q \in \mathcal{A}$ and $P \neq Q$ then $P \cap Q = \varnothing$. Let $S = \bigcup_{P \in \mathcal{A}} P$. Suppose $f : S \to [0, +\infty]$ is a function. Then

$$\sum_{s \in S} f(s) = \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s).$$

These are sums of elements of $[0, +\infty]$, and so converge in $[0, +\infty]$.
Suppose $F \subseteq S$ is finite.

$$F = \bigcup_{P \in \mathcal{B}} P \cap F.$$

where $\mathcal{B} = \{P \in \mathcal{A}, P \cap F \neq \varnothing\}$. This is a finite union of finite disjoint sets so

$$\sum_{s \in F} f(s) = \sum_{P \in \mathcal{B}} \sum_{s \in P \cap F} f(s).$$

Sums of sums (2/6)

$P \cap F \subseteq P$ and $f(s) \in [0, +\infty]$ so

$$\sum_{s \in P \cap F} f(s) \leq \sum_{s \in P} f(s)$$

and therefore

$$\sum_{P \in \mathcal{B}} \sum_{s \in P \cap F} f(s) \leq \sum_{P \in \mathcal{B}} \sum_{s \in P} f(s).$$

Also, $\mathcal{B} \subseteq \mathcal{A}$ and

$$\sum_{s \in P} f(s) \in [0, +\infty]$$

for all $P \in \mathcal{A}$ so

$$\sum_{P \in \mathcal{B}} \sum_{s \in P} f(s) \leq \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s).$$

Combining the previous results,

$$\sum_{s \in F} f(s) \leq \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s).$$

Sums of sums (3/6)

Taking limits in

$$\sum_{s \in F} f(s) \leq \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s)$$

with respect to the net of finite subsets F of S gives

$$\sum_{s \in S} f(s) \leq \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s)$$

Next we prove the reverse inequality.
Suppose that $\mathcal{G} \subseteq \mathcal{A}$ is finite. Then

$$\sum_{P \in \mathcal{G}} \sum_{s \in P} f(s) = \sum_{P \in \mathcal{G}} \sup \sum_{s \in F_P} f(s)$$

where the supremum is over finite subsets F_P of P. This is the same as

$$\sup \sum_{P \in \mathcal{G}} \sum_{s \in F_P} f(s)$$

where the supremum is over all choices of an F_P for each $P \in \mathcal{G}$.

Sums of sums (4/6)

Each choice of an F_P for each $P \in \mathcal{G}$ is uniquely determined by the set

$$H = \bigcup_{P \in \mathcal{G}} F_P,$$

which is a finite subset of S with the property that $H \cap Q = \varnothing$ if $Q \notin \mathcal{G}$, because P_F can be recovered from H by

$$F_P = H \cap P.$$

Therefore

$$\sum_{P \in \mathcal{G}} \sum_{s \in P} f(s) = \sup \sum_{P \in \mathcal{G}} \sum_{s \in H \cap P} f(s) = \sup \sum_{s \in H} f(s),$$

where the supremum is over all such finite subsets H. This is less than or equal to the supremum over all finite subsets, which is just $\sum_{s \in S} f(s)$ by definition, so

$$\sum_{P \in \mathcal{G}} \sum_{s \in P} f(s) \leq \sum_{s \in S} f(s).$$

Sums of sums (5/6)

Taking the supremum of

$$\sum_{P \in \mathcal{G}} \sum_{s \in P} f(s) \leq \sum_{s \in S} f(s)$$

over all finite subsets $\mathcal{G}$ of $\mathcal{A}$ gives

$$\sum_{P \in \mathcal{A}} \sum_{s \in P} f(s) \leq \sum_{s \in S} f(s).$$

Since we already have the reverse inequality we conclude that

$$\sum_{s \in S} f(s) = \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s).$$

This completes the proof that

$$\sum_{s \in S} f(s) = \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s).$$

Sums of sums (6/6)

Two consequences of the theorem are particularly important.

Suppose $S = P \cup Q$, $P \cap Q = \varnothing$ and $f : S \to [0 + \infty]$ is a function. Then

$$\sum_{s \in S} f(s) = \sum_{s \in P} f(s) + \sum_{s \in Q} f(s).$$

This is just the proposition above with $\mathcal{A} = \{P, Q\}$.

Suppose A and B are sets and $f : A \times B \to [0, +\infty]$ is a function. Then

$$\sum_{a \in A} \sum_{b \in B} f(a, b) = \sum_{(a,b) \in A \times B} f(a, b) = \sum_{b \in B} \sum_{a \in A} f(a, b).$$

This is known as Tonelli's Theorem and will be proved next time. That proof will be our first application of the convergence theorems for sums.