MAU22200 Lecture 38

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Limits and sums

We saw last time that

$$\lim_{n\to\infty} \sum_{s\in S} f_n(s) = \sum_{s\in S} \lim_{n\to\infty} f_n(s)$$

can fail even when all the limits and sums exist. We need theorems which tell us this equation does hold under some hypotheses on f, or that something weaker, e.g. an inequality, holds under weaker hypotheses. There are three main theorems of this type:

- The Monotone Convergence Theorem
- Fatou's Lemma
- The Dominated Convergence Theorem

Each of these will later be seen to be a special case of a theorem for integrals, but it's convenient to have the special cases earlier.

The Monotone Convergence Theorem (1/4)

Suppose S is a set and $f : \mathbf{N} \times S \to [0, +\infty]$ is a function such that if $m \leq n$ then $f_m(s) \leq f_n(s)$ for all $s \in S$. Then

$$\lim_{n\to\infty} \sum_{s\in S} f_n(s) = \sum_{s\in S} \lim_{n\to\infty} f_n(s).$$

This is called the Monotone Convergence Theorem (for sums). Because $f_m(s) \leq f_n(s)$ if $m \leq n$ the limit

$$\lim_{n\to\infty} f_n(s)$$

exists (in $[0, +\infty]$) for all $s \in S$. Because $f_n(s) \in [0, +\infty]$ the sum

$$\sum_{s\in S} f_n(s)$$

exists (in $[0, +\infty]$) for all $n \in \mathbf{N}$.

The Monotone Convergence Theorem (2/4)

From $f_m(s) \leq f_n(s)$ it follows that

$$\sum_{s\in S} f_m(s) \leq \sum_{s\in S} f_n(s).$$

$\sum_{s\in S} f_n(s)$ is therefore monotone increasing in n. So $\lim_{s\in S} \sum_{s\in S} f_n(s)$ exists (in $[0,+\infty]$).
$\lim_{n\to\infty} f_n(s) \in [0,+\infty]$ for all $s \in S$ so $\sum_{s\in S} \lim_{n\to\infty} f_n(s)$ exists (in $[0,+\infty]$).
So both sides of the equation

$$\lim_{n\to\infty} \sum_{s\in S} f_n(s) = \sum_{s\in S} \lim_{n\to\infty} f_n(s)$$

are meaningful. None of this would have worked in $[0,+\infty)$, which is why we introduced the extended reals. We still need to show both sides are equal though.

The Monotone Convergence Theorem (3/4)

$f_n(s)$ is an increasing sequence in n for each s so

$$\lim_{n\to\infty} f_n(s) = \sup_{n\in\mathbb{N}} f_n(s).$$

So

$$f_n(s) \leq \lim_{n\to\infty} f_n(s).$$

Therefore

$$\sum_{s\in S} f_n(s) \leq \sum_{s\in S} \lim_{n\to\infty} f_n(s).$$

Suppose F is a finite subset of S. Then

$$\sum_{s\in F} f_n(s) \leq \sum_{s\in S} f_n(s).$$

$$\lim_{n\to\infty} \sum_{s\in F} f_n(s) \leq \lim_{n\to\infty} \sum_{s\in S} f_n(s).$$

The Monotone Convergence Theorem (4/4)

$$\lim_{n\to\infty} \sum_{s\in F} f_n(s) \leq \lim_{n\to\infty} \sum_{s\in S} f_n(s).$$

We can interchange limits and finite sums so

$$\lim_{n\to\infty} \sum_{s\in F} f_n(s) = \sum_{s\in F} \lim_{n\to\infty} f_n(s).$$

Therefore

$$\sum_{s\in F} \lim_{n\to\infty} f_n(s) \leq \lim_{n\to\infty} \sum_{s\in S} f_n(s).$$

Taking the limit over F,

$$\sum_{s\in S} \lim_{n\to\infty} f_n(s) \leq \lim_{n\to\infty} \sum_{s\in S} f_n(s).$$

We now have the inequality in both directions.

Fatou's Lemma (1/3)

Suppose S is a set and $f : \mathbf{N} \times S \to [0, +\infty]$ is a function. Then

$$\sum_{s \in S} \sup_{m \in \mathrm{N}} \inf_{n \geq m} f_n(s) \leq \sup_{m \in \mathrm{N}} \inf_{n \geq m} \sum_{s \in S} f_n(s).$$

This is called Fatou's Lemma. Define

$$g_m(s) = \inf_{n \geq m} f_n(s).$$

If $l \leq m$ then $g_l(s) \leq g_m(s)$. In other words, $g_m(s)$ is a monotone function of m for each $s \in S$. Also, $\sum_{s \in S} g_l(s) \leq \sum_{s \in S} g_m(s)$, so $\sum_{s \in S} g_m(s)$ is a monotone function of m. Therefore

$$\lim_{m \to \infty} g_m(s) = \sup_{m \in \mathrm{N}} g_m(s)$$

$$\lim_{m \to \infty} \sum_{s \in S} g_m(s) = \sup_{m \in \mathrm{N}} \sum_{s \in S} g_m(s).$$

Fatou's Lemma (2/3)

By the Monotone Convergence Theorem

$$\lim_{m\to\infty} \sum_{s\in S} g_m(s) = \sum_{s\in S} \lim_{m\to\infty} g_m(s).$$

$$\sup_{m\in \mathrm{N}} \sum_{s\in S} g_m(s) = \sum_{s\in S} \sup_{m\in \mathrm{N}} g_m(s).$$

If $l \leq m$ then

$$g_l(s) = \inf_{n\geq l} f_n(s) \leq f_m(s).$$

so

$$\sum_{s\in S} g_l(s) \leq \sum_{s\in S} f_m(s).$$

Taking the infimum over $m \geq l$ gives

$$\sum_{s\in S} g_l(s) \leq \inf_{l\leq m} \sum_{s\in S} f_m(s).$$

Fatou's Lemma (3/3)

$$\sum_{s\in S} g_m(s) \le \inf_{m\ge n} \sum_{s\in S} f_n(s).$$

$$\sup_{m\in \mathrm{N}} \sum_{s\in S} g_m(s) \le \sup_{m\in \mathrm{N}} \inf_{m\ge n} \sum_{s\in S} f_n(s).$$

$$\sum_{s\in S} \sup_{m\in \mathrm{N}} g_m(s) \le \sup_{m\in \mathrm{N}} \inf_{m\ge n} \sum_{s\in S} f_n(s).$$

$$\sum_{s\in S} \sup_{m\in \mathrm{N}} \inf_{n\ge m} f_n(s) \le \sup_{m\in \mathrm{N}} \inf_{m\ge n} \sum_{s\in S} f_n(s).$$

So we're done. We saw that $\inf_{n\ge m} f_n(s)$ is monotone in m so $\sup_{m\in \mathrm{N}} \inf_{n\ge m} f_n(s)$ is the same as $\lim_{m\to\infty} \inf_{n\ge m} f_n(s)$. That's why this is usually written as $\liminf_{n\to\infty} f_n(s)$. Fatou's Lemma is usually written in terms of $\liminf$ rather than $\sup\inf$ but for our purposes $\sup\inf$ is a more useful way to think about it.

A useful lemma

For sequences in **R** it's true that

$$\sup_{m\in\mathrm{N}} \inf_{n\geq m} \varphi_n \leq \inf_{m\in\mathrm{N}} \sup_{n\geq m} \varphi_n$$

provided both sides exist. Furthermore, if

$$\inf_{m\in\mathrm{N}} \sup_{n\geq m} \varphi_n \leq \sup_{m\in\mathrm{N}} \inf_{n\geq m} \varphi_n$$

then $\lim_{n\to\infty} \varphi_n$ exists and

$$\sup_{m\in\mathrm{N}} \inf_{n\geq m} \varphi_n = \lim_{n\to\infty} \varphi_n = \inf_{m\in\mathrm{N}} \sup_{n\geq m} \varphi_n.$$

This can be improved in two ways: We can replace sequences by nets. We can replace **R** by any interval in $[-\infty, +\infty]$. See the notes for details. Also for any example of how to deal with $[-\infty, +\infty]$ without case by case analysis.