MAU22200 Lecture 37

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Convergence and absolute convergence

A series $\sum_{j=0}^{\infty} \xi_j$ (in $\mathbf{R}$) is called *convergent* if the sequence of partial sums $\sigma_n = \sum_{j=0}^{n} \xi_j$ converges (in $\mathbf{R}$). The series is called *absolutely convergent* if the sequence of sums of absolute values $\alpha_n = \sum_{j=0}^{n} |\xi_j|$ converges (in $\mathbf{R}$).
The words "in $\mathbf{R}$" were previously redundant, but the sequence α is a monotone sequence in $[0, +\infty]$ and so always converges in $[0, +\infty]$. It converges absolutely in $\mathbf{R}$ if and only if

$$\sum_{j=0}^{\infty} |\xi_j| < +\infty.$$

Every absolutely convergent series is convergent. Ultimately this follows from completeness. It isn't true in $\mathbf{Q}$. Not every convergent series is absolutely convergent. $\sum_{j=0}^{\infty} (-1)^j/(j+1)$ is convergent, but not absolutely convergent.

Sums

If S is a set and $f : S \to \mathbf{R}$ is a function then the sum $\sum_{s\in S} f(s)$ is said to be *convergent* (in $\mathbf{R}$) if the net of partial sums $\sum_{s\in F} f(s)$ is convergent (in $\mathbf{R}$), where F ranges over the finite subsets of S, ordered by inclusion. $\sum_{s\in S} f(s)$ is said to be *absolutely convergent* (in $\mathbf{R}$) if the net of partial sums $\sum_{s\in F} |f(s)|$ is convergent (in $\mathbf{R}$). The net of these partial sums is monotone, so $\sum_{s\in S} f(s)$ is always absolutely convergent in $[0, +\infty]$. It's convergent in $\mathbf{R}$ if and only if

$$\sum_{s\in S} |f(s)| < +\infty.$$

Every absolutely convergent sum is convergent, just as for series. Every convergent sum is absolutely convergent, unlike for series!

The Comparison Test

Every absolutely convergent sum is convergent. This is a consequence of the comparison test.

> *Suppose that u and v are functions from a set S to* $\mathbf{R}$*. If* $|u(s)| \leq |v(s)|$ *and* $\sum_{s \in S} |v(s)| < +\infty$ *then* $\sum_{s \in S} u(s)$ *converges in* $\mathbf{R}$*.*

This is analogous to the comparison test for series. The statement at the top is the special $u = v$.
Here's a strategy for the proof of the proposition: The condition $\sum_{s \in S} |v(s)| < +\infty$ is equivalent to convergence in $\mathbf{R}$. Convergence of a sum is defined to mean convergence of the net of partial sums over finite subsets. Convergence of nets in $\mathbf{R}$ is equivalent to the Cauchy condition. So we show that if the net of sums of absolute values of v over finite subsets is Cauchy then so is the net of sums of u over finite subsets.

Proof of Comparison Test (1/3)

Suppose then that for every $\epsilon > 0$ there is a finite F such that if $F \subseteq G$ and $F \subseteq H$ then

$$\left| \sum_{s \in G} |v(s)| - \sum_{s \in H} |v(s)| \right| < \epsilon.$$

This is just the Cauchy condition for $\sum_{s \in S} |v(s)|$. It applies to $H = F$, since $F \subseteq F$.

$$\left| \sum_{s \in G} |v(s)| - \sum_{s \in F} |v(s)| \right| < \epsilon.$$

$$\sum_{s \in G} |v(s)| - \sum_{s \in F} |v(s)| = \sum_{s \in G \setminus F} |v(s)| \geq 0$$

because $F \subseteq G$ and $|v(s)| \geq 0$.

Proof of Comparison Test (2/3)

$$\sum_{s \in G} u(s) - \sum_{s \in F} u(s) = \sum_{s \in G \setminus F} u(s) \leq \sum_{s \in G \setminus F} |v(s)| < \epsilon$$

since $u(s) \leq |u(s)| \leq |v(s)|$. The same argument applied to $u(s) \geq -|u(s)| \geq -|v(s)|$ gives

$$\sum_{s \in G} u(s) - \sum_{s \in F} u(s) > -\epsilon.$$

So if $F \subseteq G$ then

$$\left| \sum_{s \in G} u(s) - \sum_{s \in F} u(s) \right| < \epsilon.$$

Similarly, if $F \subseteq H$ then

$$\left| \sum_{s \in H} u(s) - \sum_{s \in F} u(s) \right| < \epsilon.$$

Proof of Comparison Test (3/3)

If both $F \subseteq G$ and $F \subseteq H$ then

$$\left| \sum_{s \in G} u(s) - \sum_{s \in H} u(s) \right| < 2\epsilon.$$

For every $\epsilon > 0$ there is an F with this property, which is the Cauchy condition for $\sum_{s \in S} u(s)$, except with 2ϵ in place of ϵ, but that's easily dealt with. So we're done.

The converse (1/2)

The converse, that $\sum_{s \in S} |u(s)| < +\infty$ if $\sum_{s \in S} u(s)$ converges, is more interesting.

We use the Cauchy condition again. For any $\epsilon > 0$ there is a finite $F \subseteq S$ such that if G, H are finite subsets of S satisfying $F \subseteq G$ and $F \subseteq H$ then

$$\left| \sum_{s \in G} u(s) - \sum_{s \in H} u(s) \right| < \epsilon.$$

For any finite $K \subseteq S$ we choose

$$G = \{s \in F \cup K : s \in F \text{ or } u(s) > 0\},$$
$$H = \{s \in F \cup K : s \in F \text{ or } u(s) < 0\}.$$

Then G, H have the properties listed above and

$$\sum_{s \in G} u(s) - \sum_{s \in H} u(s) = \sum_{s \in K \setminus F} |u(s)|.$$

The converse (2/2)

$$\sum_{s \in K \setminus F} |u(s)| < \epsilon$$

So

$$\sum_{s \in K} |u(s)| = \sum_{s \in K \cap F} |u(s)| + \sum_{s \in K \setminus F} |u(s)| \leq \sum_{s \in F} |u(s)| + \epsilon.$$

Choose any $\epsilon > 0$ and the corresponding F. The inequality above then holds for all K. Taking the limit over all finite subsets K gives

$$\sum_{s \in S} |u(s)| \leq \sum_{s \in F} |u(s)| + \epsilon < +\infty.$$

So we're done.

Convergence theorems (1/2)

The next question we need to address is whether we can exchange sums and limits. Consider $S = \mathbf{N}$ and

$$f_n(s) = \begin{cases} 1 & \text{if } s = n, \\ 0 & \text{if } s \neq n. \end{cases}$$

$$\lim_{n\to\infty} f_n(s) = 0$$

for every s because $f_n(s) = 0$ for all $n > s$. So

$$\sum_{s\in S} \lim_{n\to\infty} f_n(s) = 0.$$

$$\sum_{s\in S} f_n(s) = 1$$

for all $n \in \mathbf{N}$. So

$$\lim_{n\to\infty} \sum_{s\in S} f_n(s) = 1$$

Convergence theorems (1/2)

So

$$\lim_{n\to\infty} \sum_{s\in S} f_n(s) \neq \sum_{s\in S} \lim_{n\to\infty} f_n(s)$$

in this example, even though all the sums and limits converge (in $\mathbf{R}$). Next time we'll prove

$$\lim_{n\to\infty} \sum_{s\in S} f_n(s) = \sum_{s\in S} \lim_{n\to\infty} f_n(s)$$

under various hypotheses on f. Although we mostly need those results for sequences we will sometimes need them for nets, so I've proven them in that context in the notes. In lecture I'll just do the special case of sequences.