MAU22200 Lecture 36

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Extending the reals

The Banach-Tarski paradox from last lecture has nothing to do with sets which are too big to attach a finite volume to–all the sets involved are bounded subsets of $\mathbf{R}^3$–but with sets which are too ugly. But we still need to consider sets which are too big to have a finite volume. For this it's useful to work with the *extended reals* $[-\infty, +\infty]$ or the extended non-negative reals $[0, +\infty]$.

$$[-\infty, +\infty] = \{-\infty\} \cup \mathbf{R} \cup \{+\infty\}, \qquad [0, +\infty] = [0, +\infty) \cup \{+\infty\}.$$

We can extend some, but not all, of the structure of the reals to the extended reals. For example, we extend the order relation by

$$-\infty < x < +\infty$$

for all $x \in \mathbf{R}$.

Bounds, infima and suprema

In $\mathbf{R}$ some subsets have upper or lower bounds and some don't. For example $(0, 1)$ has both an upper and lower bound while $\mathbf{Z}$ has neither. In $[-\infty, +\infty]$ *every* subset has an upper and a lower bound. $+\infty$ is an upper bound and $-\infty$ is a lower bound.
In $\mathbf{R}$ every non-empty bounded subset has an infimum and a supremum. In $[-\infty, +\infty]$ *every* subset has an infimum and a supremum. For example,

$$\inf \varnothing = +\infty, \qquad \sup \varnothing = -\infty$$

It's in this sense that we can say $\inf_{s \in K, t \in C} d(s, t) > 0$ when $C = \varnothing$ or $K = \varnothing$ in Lecture 34.
It's not obvious that every subset has an infimum and supremum. The proof in the notes is fairly long and is based on case by case analysis. It relies on the corresponding theorem for $\mathbf{R}$.

Bounds, infima and suprema, continued

It's convenient for every subset to have an infimum and supremum, but we pay a (small) price.

$$\inf A \leq \sup A$$

is now true only for non-empty $A \subseteq [-\infty, +\infty]$. For $A = \varnothing$ it's false because $\inf \varnothing = +\infty$ and $\sup \varnothing = -\infty$. If $A \neq \varnothing$ then there's an $x \in A$ and we have $\inf A \leq x \leq \sup A$.
Intervals are described in the way you might expect, e.g. $[a, b) = \{x \in [-\infty, +\infty] \colon a \leq x < b\}$ even if a or b is infinite. This is consistent with the notation from $\mathbf{R}$. It's also consistent with the notation $[-\infty, +\infty]$.

Arithmetic (1/2)

We can also extend the usual arithmetic operations.

$$\begin{aligned} w + +\infty = +\infty + w &= +\infty, & x + -\infty = -\infty + x &= -\infty, \\ y \cdot +\infty = +\infty \cdot y &= +\infty, & y \cdot -\infty = -\infty \cdot y &= -\infty, \\ z \cdot +\infty = +\infty \cdot z &= -\infty, & z \cdot -\infty = -\infty \cdot z &= +\infty, \end{aligned}$$

for $w \in (-\infty, +\infty]$, $x \in [-\infty, +\infty)$, $y \in (0, +\infty]$ and $z \in [-\infty, 0)$. Also,

$$\begin{aligned} +\infty - x &= +\infty, & w - -\infty &= +\infty, \\ -\infty - w &= -\infty, & x - +\infty &= -\infty. \end{aligned}$$

Somewhat reluctantly, we also define

$$0 \cdot +\infty = +\infty \cdot 0 = 0 = 0 \cdot -\infty = -\infty \cdot 0.$$

Arithmetic (2/2)

Some sums and differences are deliberately left undefined, e.g. $+\infty - +\infty$.

The associative, commutative and distributive laws hold, provided all the relevant sums and products are defined.

Cancellation laws can fail though. From $x + y = x + z$ it no longer follows that $y = z$. A counterexample is $+\infty + 0 = +\infty = +\infty + 1$ but $0 \neq 1$. $x + y = x + z \Rightarrow y = z$ is proved by subtracting x from both sides, which is fine in $\mathbf{R}$ but here the relevant differences are undefined.

Topology

The topology on $[-\infty, +\infty]$ is the one generated by intervals of the form $(a, +\infty]$ and $[-\infty, b)$, where $a, b \in [-\infty, +\infty]$. This choice makes statements like $\lim_{n\to\infty} \alpha_n = +\infty$ have the expected meaning, i.e. that for all $K > 0$ there is an m such that if $n \geq m$ then $\alpha_n > K$.

Addition is continuous with respect to this topology.

Multiplication is continuous except at the points $(0, +\infty)$, $(0, -\infty)$, $(+\infty, 0)$ and $(-\infty, 0)$. It can't be continuous at $(0, +\infty)$ because

$$\lim_{n\to\infty} \frac{1}{n} \cdot n = \lim_{n\to\infty} 1 = 1 \neq 0 = 0 \cdot +\infty = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} n.$$

There are similar examples for the other points of discontinuity.

This is why I said I was reluctant to define

$$0 \cdot +\infty = +\infty \cdot 0 = 0 = 0 \cdot -\infty = -\infty \cdot 0.$$

Monotone sequences (and nets)

Now that we have a topology we can talk about convergence and limits. Bounded monotone sequences in **R** converge. Actually the same is true of nets. The limit is equal to the supremum, for increasing sequences or nets, or to the infimum, for decreasing sequences or nets.
For sequences (or nets) in $[-\infty, +\infty]$ we can drop the boundedness assumption. This isn't obvious. The proof is again a case by case analysis.

Sums

The extended reals may look unfamiliar but you've probably seen them informally. Statements like

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$$

are only really meaningful in the extended reals. Even in the extended reals, not all sums exist. $\sum_{n=1}^{\infty} (-1)^n$ does not exist, even as an extended real number. The partial sums oscillate between -1 and 0.

Sums of non-negative terms, i.e. of elements of $[0, +\infty]$, on the other hand, *always* exist as elements of $[0, +\infty]$. The proof is easy, given the convergence of monotone nets proved earlier. You take the directed set to be the set of finite subsets and the net to be the function which assigns to each such subset the corresponding partial sum.

When and how to use the extended reals

The extended reals aren't a universal replacement for the usual reals. Some types of argument don't work there, e.g. convergence isn't equivalent to the Cauchy criterion. Relatedly, there needn't be an element $x \in A$ such that $x > \sup A - \epsilon$ or one such that $x < \inf A + \epsilon$. Often it's useful to go back and forth between the extended reals and the usual ones. For example, we might first show that a sequence converges in the extended reals, then show that the limit is finite, so it actually converges in the usual reals, and then conclude that it's Cauchy. Using the extended reals generally doesn't allow us to avoid questions of finiteness, but it can allow us to postpone them. For area/volume it allows you to show a set isn't too ugly before worrying about whether it's too big. For sums it allows you to show the sum isn't too oscillatory before worrying about whether it's unbounded.