MAU22200 Lecture 34

John Stalker

Trinity College Dublin

24 January 2022

Problem solving suggestions

Marking the Schol exams convinced me that I should talk more about problem solving.
Here's on of the problems from this year's paper, which illustrates some useful points:

1. *Suppose* (X, d) *is a metric space,* $C \subseteq X$ *is closed,* $K \subseteq X$ *is compact, and* $C \cap K = \varnothing$*. Show that* $\inf_{s \in C, t \in K} d(s,t) > 0$.
2. *Show that the hypothesis that* K *is compact cannot be replaced by the the hypothesis that* K *is closed. In other words, give an example of a metric space* (X, d) *and closed subsets* C *and* K *of* X *such that* $C \cap K = \varnothing$ *but* $\inf_{s \in C, t \in K} d(s,t) = 0$.

If you didn't do the Schol this year then I'd suggest pausing the video and taking an hour or so to try the problem yourself before listening to my comments.

Using all the parts of the problem

My first point is *don't consider the parts of question in isolation* unless they're really separate.

From the statement of the second part you know you'll need to use the compactness of K. If you think you have a proof which doesn't use compactness then one of us has made a mistake.

From the statement of the first part you know that your example in the second part should have two non-compact sets. If it doesn't then one of us has made a mistake.

Closed bounded subsets of $\mathbf{R}^n$ are compact, and the example you're asked for in the second part has closed sets. If your example is in $\mathbf{R}^n$ then both subsets must be unbounded. This is useful for checking your answer, but also in trying to find an answer.

An incorrect answer to Part 1

Here's an invalid proof for the first part:

> *$C \cap K = \varnothing$ so if $s \in C$ and $t \in K$ then $s \neq t$. d is a metric, so therefore $d(s,t) > 0$. It follows that $\inf_{s \in C, t \in K} d(s,t) > 0$.*

The last sentence is wrong because the infimum of a set of positive numbers needn't be positive. Even if you don't immediately see *why* it's wrong you should see that it must be wrong because it doesn't require K to be compact. If this were a valid proof then there could be no correct answer to the second part.

Beware of trivial cases!

I should have stated the first part of the problem as

> *Suppose (X, d) is a metric space, $C \subseteq X$ is non-empty and closed, $K \subseteq X$ is non-empty and compact, and $C \cap K = \varnothing$. Show that $\inf_{s \in C, t \in K} d(s, t) > 0$.*

Do you see the difference between this and the previous version? Here I've assumed C and K are non-empty. If I don't do that then the set of numbers $d(s, t)$ with $s \in C$, $t \in K$ is empty. Is the statement still correct in that case? That depends on your definitions. There are actually two sets of conventions. According to one $\inf_{s \in C, t \in K} d(s, t) > 0$ is still true. According to the other it's meaningless. We'll return to this question later this week. Luckily (?) it didn't lead anyone astray on the exam.

Follow your nose

We know we need to use the fact that K is compact. Compact sets are those for which every open cover has a finite subcover. We probably want to use the fact that C is closed, i.e. $X \setminus C$ is open, i.e. for every $x \in X \setminus C$ there is an $r > 0$ such that $B(x, r) \subseteq X \setminus C$.
And we probably want to use the fact that $C \cap K = \emptyset$, i.e. $K \subseteq X \setminus C$.
The sets $B(x, r)$ such that $x \in K$ and $B(x, r) \subseteq X \setminus C$ are an open cover of K, and so have a finite subcover. In other words, there are $x_1, \ldots, x_m$ and $r_1, \ldots, r_m$ such that if $y \in K$ then $d(y, x_j) < r_j$ for some j and if $z \in C$ then $d(z, x_j) \geq r_j$ for all j.

$$d(y, z) \geq |d(z, x_j) - d(y, x_j)| = d(z, x_j) - d(y, x_j) > 0.$$

The limits of following your nose

We've just done the more or less obvious thing at each stage and found that if $y \in K$ and $z \in C$ then $d(y,z) > 0$, which we already knew, and which we know isn't sufficient. So "follow your nose" doesn't give a solution. That's why this was a Schol question. Can this be improved? There are various tricks you've seen in proofs in the notes, one of which works here. The sets $B(x, r/2)$ where $x \in K$ and $B(x,r) \subseteq X \setminus C$ also form an open cover of K. What's new is the "/2". Take a finite subcover. There are $x_1, \ldots, x_m$ and $r_1, \ldots, r_m$ such that if $y \in K$ then $d(y, x_j) < r_j/2$ for some j and if $z \in C$ then $d(z, x_j) \geq r_j$ for all j.

$$d(y,z) \geq |d(z,x_j) - d(y,x_j)| = d(z,x_j) - d(y,x_j) > \frac{r_j}{2} \geq \min_{1 \leq j \leq m} \frac{r_j}{2}.$$

$\min_{1 \leq j \leq m} r_j/2$ is a positive lower bound for the set of $d(y,z)$ with $y \in K$ and $z \in C$, so the infimum is positive.

Theorems are usually better than definitions

The proof above works, but it's not the shortest proof. It used only the definitions, so there's certain amount of wheel reinvention involved. To find a shorter proof, look for a theorem with similar hypotheses or conclusions to what you want.

> *Suppose (X, d) is a metric space and $A \in \wp(X)$ is non-empty. Define $r \colon X \to \mathbf{R}$ by*
>
> $$r(x) = \inf_{y \in A} d(x, y).$$
>
> *Then $r(x) \geq 0$ for all x, $r(x) = 0$ if and only if $x \in \overline{A}$ and r is Lipschitz continuous.*

We can apply this to $A = C$. Set $r(x) = \inf_{y \in C} d(x, y)$. r is a (Lipschitz) continuous function and $r(x) > 0$ for $x \notin \overline{C} = C$, so for $x \in K$.

Finishing the proof

$r(x) = \inf_{y \in C} d(x, y)$ is a (Lipschitz) continuous function and $r(x) > 0$ for $x \in K$. What do we know about continuous functions on non-empty compact sets? They have minima and maxima! So there is a $w \in K$ such that

$$r(w) = \inf_{x \in K} r(x) = \inf_{x \in K} \inf_{y \in C} d(x, y) = \inf_{x \in K, y \in C} d(x, y).$$

$w \in K$ so $r(w) > 0$ and hence $\inf_{x \in K, y \in C} d(x, y) > 0$, as required.
This uses a number of theorems: the one quoted above, the Extreme Value Theorem, and the fact that Lipschitz continuity implies continuity. It's shorter than the other proof though, and doesn't require any clever tricks.

The second part

> *Show that the hypothesis that K is compact cannot be replaced by the the hypothesis that K is closed. In other words, give an example of a metric space (X, d) and closed subsets C and K of X such that $C \cap K = \varnothing$ but $\inf_{s \in C, t \in K} d(s, t) = 0$.*

There's no procedure for generating examples like this, but you do have some information. K must not be compact, so if your example is in $\mathbf{R}^n$ then it must be unbounded, e.g.

$$C = \{(x, y) \in \mathbf{R}^2 \colon xy \geq 1\}, \qquad K = \{(x, y) \in \mathbf{R}^2 \colon xy \leq -1\}.$$

You don't have to choose $X = \mathbf{R}^n$ though. Here's a nice example:

$$X = \mathbf{R} \setminus \{0\}, \qquad C = [-1, 0), \qquad K = (0, 1].$$

C and K are closed and bounded, but not compact. Heine-Borel fails, even though we removed only a single point from $\mathbf{R}$.