# MAU22200 Lecture 33

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# Equivalence of norms on finite dimensional spaces (1/4)

Suppose $V$ is finite dimensional and $p$ is a norm on $V$.

$$p(\mathbf{w}) \leq p(\mathbf{w}-\mathbf{z}) + p(\mathbf{z})$$

and

$$p(\mathbf{z}) \leq p(\mathbf{z}-\mathbf{w}) + p(\mathbf{w})$$

$$-p(\mathbf{z}-\mathbf{w}) \leq p(\mathbf{w}) - p(\mathbf{z}) \leq p(\mathbf{w}-\mathbf{z})$$

$$|p(\mathbf{w}) - p(\mathbf{z})| \leq p(\mathbf{w}-\mathbf{z}).$$

Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be a basis for $V$. Define $\mathbf{f} \colon \mathbf{R}^n \to V$ by

$$\mathbf{f}(\mathbf{x}) = \sum_{j=1}^{n} x_j \mathbf{u}_j, \qquad g(\mathbf{x}) = p(\mathbf{f}(\mathbf{x})).$$

# Equivalence of norms on finite dimensional spaces (2/n)

$$
\begin{aligned}
|g(\mathbf{x}) - g(\mathbf{y})| &= |p(\mathbf{f}(\mathbf{x})) - p(\mathbf{f}(\mathbf{y}))| \leq p(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})) \\
&= p(\mathbf{f}(\mathbf{x} - \mathbf{y})) = p\left(\sum_{j=1}^{n} (x_j - y_j)\mathbf{u}_j\right) \\
&\leq \sum_{j=1}^{n} |x_j - y_j| p\left(\mathbf{u}_j\right) \\
&\leq \sqrt{\sum_{j=1}^{n} |x_j - y_j|^2} \sqrt{\sum_{j=1}^{n} p\left(\mathbf{u}_j\right)^2} \\
&= K\|\mathbf{x} - \mathbf{y}\|
\end{aligned}
$$

where $K = \sqrt{\sum_{j=1}^{n} p\left(\mathbf{u}_j\right)^2}$. The norm in $\|\mathbf{x} - \mathbf{y}\|$ is the usual Euclidean norm on $\mathbf{R}^n$. So $g$ is Lipschitz, hence continuous.

# Equivalence of norms on finite dimensional spaces (3/4)

Let $S = \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x}\| = 1\}$. If $\mathbf{x} \in S$ then $\mathbf{x} \neq \mathbf{0}$ and hence $\mathbf{f}(\mathbf{x}) = \sum_{j=1}^n x_j \mathbf{u}_j$ is a non-trivial linear combination of basis vectors of $V$. So $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ and therefore $g(\mathbf{x}) = p(\mathbf{f}(\mathbf{x})) > 0$.
Similarly if $q$ is a norm on $V$ and $h(\mathbf{x}) = q(\mathbf{f}(\mathbf{x}))$ then $h$ is continuous and is positive on $S$. $q/p$ is therefore a continuous positive function on $S$.
$S$ is closed and bounded, hence compact, so $h/g$ has a minimum and maximum on $S$, both of which must both be positive. There are therefore $c, C > 0$ such that

$$c \leq \frac{h(\mathbf{x})}{g(\mathbf{x})} \leq C$$

for all $\mathbf{x} \in S$. If $\mathbf{y} \neq \mathbf{0}$ then

$$\mathbf{x} = \frac{1}{\|\mathbf{y}\|}\mathbf{y}$$

is an element of $S$.

# Equivalence of norms on finite dimensional spaces (4/4)

$$c \leq \frac{h(\mathbf{x})}{g(\mathbf{x})} \leq C$$

and

$$\begin{aligned}\frac{q(\mathbf{f}(\mathbf{y}))}{p(\mathbf{f}(\mathbf{y}))} &= \frac{q(\mathbf{f}(\|\mathbf{y}\|\mathbf{x}))}{p(\mathbf{f}(\|\mathbf{y}\|\mathbf{x}))} = \frac{q(\|\mathbf{y}\|\mathbf{f}(\mathbf{x}))}{p(\|\mathbf{y}\|\mathbf{f}(\mathbf{x}))} \\ &= \frac{\|\mathbf{y}\|q(\mathbf{f}(\mathbf{x}))}{\|\mathbf{y}\|p(\mathbf{f}(\mathbf{x}))} = \frac{h(\mathbf{x})}{g(\mathbf{x})}\end{aligned}$$

so

$$cp(\mathbf{f}(\mathbf{y})) \leq q(\mathbf{f}(\mathbf{y})) \leq Cp(\mathbf{f}(\mathbf{y})).$$

This was proved for $\mathbf{y} \neq \mathbf{0}$ but clearly also holds for $\mathbf{y} = \mathbf{0}$. $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a basis for $V$ so if $\mathbf{v} \in V$ then $\mathbf{v} = \sum_{j=1}^{n} y_j \mathbf{u}_j$ for some $y_1, \ldots, y_n$. In other words, $\mathbf{v} = \mathbf{f}(\mathbf{y})$ for some $\mathbf{y}$. Therefore

$$cp(\mathbf{v}) \leq q(\mathbf{v}) \leq Cp(\mathbf{v}).$$

So $p$ and $q$ are equivalent.

# The spaces $\ell^p(\mathbf{N})$

Suppose $p \in [1, +\infty)$. Let $\ell^p(\mathbf{N})$ be the set of functions $\alpha \colon \mathbf{N} \to \mathbf{R}$, i.e. sequences in $\mathbf{R}$, such that

$$\sum_{j=0}^{\infty} |\alpha_j|^p$$

is convergent.
Then

$$\|\alpha\|_p = \left( \sum_{j=0}^{\infty} |\alpha_j|^p \right)^{1/p}$$

is a norm on $\ell^p(\mathbf{N})$. The first two properties of norms are clear, but the third is not. It is called *Minkowski's inequality* and a proof will be given in the notes. It's also complete, though this is even less clear, and so $\ell^p(\mathbf{N})$ is a Banach space.

# Inclusions

Suppose $1 \leq p \leq q < +\infty$ and $\alpha \in \ell^p(\mathbf{N})$. If $\alpha \neq 0$ then set $\beta_j = \alpha_j / \|\alpha\|_p$. Then

$$\sum_{j=0}^{\infty} |\beta_j|^p = \sum_{j=0}^{p} |\alpha_j|^p / \|\alpha\|_p = \|\alpha\|_p / \|\alpha\|_p = 1.$$

Each summand is non-negative so $|\beta_j|^p \leq 1$. It follows that

$$|\beta_j|^q = (|\beta_j|^p)^{q/p} \leq |\beta_j|^p$$

Multiplying by $\|\alpha\|_p^q$,

$$|\alpha_j|^q \leq |\alpha_j|^p \|\alpha\|_p^{p-q}$$

By the comparison test $\sum_{j=0}^{\infty} |\alpha_j|^q$ is convergent and $\|\alpha\|_q^q \leq \|\alpha\|_p^p \|\alpha\|_p^{p-q}$ This also holds if $\alpha = 0$. In other words,

$$\ell^p(\mathbf{N}) \subseteq \ell^q(\mathbf{N}), \qquad , \|\alpha\|_q \leq \|\alpha\|_p.$$

# Strict inclusions

Define $\gamma\colon \mathbf{N} \to \mathbf{R}$ by $\gamma_j = 2^{-n/r}$ if $2^n \leq j < 2^{n+1}$. Then

$$\sum_{j=0}^{\infty} |\gamma_j|^q = \sum_{n=0}^{\infty} 2^n 2^{-nq/r} = \sum_{n=0}^{\infty} 2^{-n(q-r)/r}.$$

If $q > r$ then this geometric series converges. If $q \leq r$ then it doesn't. So $\gamma \in \ell^q(\mathbf{N})$ exactly for $q > r$ If $p < q$ then we therefore have a strict inclusion

$$\ell^p(\mathbf{N}) \subset \ell^q(\mathbf{N})$$

The inclusion function $i\colon \ell^p(\mathbf{N}) \to \ell^q(\mathbf{N})$ is a continuous injection whose image is a proper (linear) subspace. It's continuous because $\|\alpha\|_q \leq \|\alpha\|_p$, so $K = 1$ is a bound.

# Density (1/2)

Suppose $\alpha \in \ell^q(\mathbf{N})$. Define $\alpha^{[k]} \in \ell^q(\mathbf{N})$ by

$$\alpha_j^{[k]} = \begin{cases} \alpha_j & \text{if } j < k, \\ 0 & \text{if } j \geq k. \end{cases}$$

$$\alpha_j - \alpha_j^{[k]} = \begin{cases} \alpha_j & \text{if } j \geq k, \\ 0 & \text{if } j < k. \end{cases}$$

So

$$\|\alpha - \alpha^{[k]}\|_q^q = \sum_{j=k}^{\infty} |\alpha_j|^q.$$

This tends to zero as $k$ tends to infinity, because $\sum_{j=0}^{\infty} |\alpha_j|^q$ is convergent, so

$$\lim_{k\to\infty} \|\alpha - \alpha^{[k]}\|_q^q = 0$$

from which it follows that $\|\alpha - \alpha^{[k]}\|_q \to 0$ and $\alpha^{[k]} \to \alpha$.

# Density (2/2)

Let $F$ be the subset of $\ell^q(\mathbf{N})$ consisting of sequences with only finitely many non-zero elements. $\alpha^{[k]} \in F$ for all $k$ and $\alpha^{[k]} \to \alpha$ so $\alpha \in \overline{F}$. $\alpha \in \ell^q(\mathbf{N})$ was arbitrary, so $\overline{F} = \ell^q(\mathbf{N})$. In other words, $F$ is dense in $\ell^q(\mathbf{N})$ for all $q \geq 1$.
If $1 \leq p < q$ then

$$F \subseteq \ell^p(\mathbf{N}) \subseteq \ell^q(\mathbf{N})$$

so $\ell^p(\mathbf{N})$ is also a dense proper (linear) subspace of $\ell^q(\mathbf{N})$. If your intuition is based on finite dimensional normed spaces then it can be hard to imagine a dense proper subspace!

# Miscellaneous weird properties of $\ell^p(\mathbf{N})$

- Closed balls in $\ell^p$ are not compact for any $p$.
- If $i\colon \ell^p(\mathbf{N}) \to \ell^q(\mathbf{N})$ for $1 \leq p < q$ is the inclusion $i(\alpha) = \alpha$ then $i_*(B)$ is however compact in $\ell^q(\mathbf{N})$ for every closed ball $B$ in $\ell^p(\mathbf{N})$.
- $\| \quad \|_p$ and $\| \quad \|_q$ are inequivalent norms on $\ell^p(\mathbf{N})$ if $1 \leq p < q$. In fact all the (uncountably many) norms $\| \quad \|_r$ for $p \leq r \leq q$ are inequivalent.
- The inclusion $i\colon \ell^p(\mathbf{N}) \to \ell^q(\mathbf{N})$ is an injection for $1 \leq p < q$, but has no bounded left inverse.