MAU22200 Lecture 32

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Normed vector spaces

Recall that a *norm* on a vector space V is a function $p\colon V \to \mathbf{R}$ such that

- $p(\mathbf{v}) = 0$ for all $\mathbf{v} \in V$ and $p(\mathbf{v}) > 0$ unless $\mathbf{v} = 0$.
- $p(\alpha \mathbf{v}) = |\alpha| p(\mathbf{v})$ for all $\alpha \in \mathbf{R}$ and $\mathbf{v} \in V$.
- $p(\mathbf{v} + \mathbf{w}) \leq p(\mathbf{v}) + p(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in V$.

If p is a norm on V then $d(\mathbf{x}, \mathbf{y}) = p(\mathbf{x} - \mathbf{y})$ is a metric on V.
Unless otherwise specified, we always consider (V, p) with this metric and the topology of open sets with respect to the metric.
All the properties of topological spaces and metric spaces apply to normed spaces, but there are some new concepts which make sense only for normed spaces.
When there's only one norm we usually write it as $\|\mathbf{v}\|$ rather than $p(\mathbf{v})$. This notation is also used when there's more than one norm being considered, but it requires care.

Equivalent norms

Two norms p and q are called *equivalent* if there are $c, C > 0$ such that

$$cp(\mathbf{v}) \leq q(\mathbf{v}) \leq Cp(\mathbf{v})$$

for all $\mathbf{v} \in V$. It's straightforward to check that equivalence is indeed and equivalence relation.

If p and q are distinct, but equivalent, norms then they give rise to distinct metrics, but the same topology. Suppose $U \in \mathcal{T}_p$. In other words, for each $\mathbf{x} \in U$ there is an $r > 0$ such that if $p(\mathbf{y} - \mathbf{x}) < r$ then $\mathbf{y} \in U$. Let $\delta = cr$. If $q(\mathbf{y} - \mathbf{x}) < \delta$ then $p(\mathbf{y} - \mathbf{x}) < \frac{1}{c} q(\mathbf{y} - \mathbf{x}) < r$. For each $\mathbf{x} \in U$ there is therefore a q-ball about $\mathbf{x}$ contained in U. Therefore $U \in \mathcal{T}_q$.

Similarly, if $U \in \mathcal{T}_q$ then $U \in \mathcal{T}_p$. So $\mathcal{T}_p = \mathcal{T}_q$.

For any notions defined purely in terms of the topology, e.g. continuity, compactness, connectedness, etc., we can therefore replace the given norm by any equivalent norm.

Banach spaces

A normed vector space (V, p) is said to be a *Banach space* if (V, d) is a complete metric space, where d is the metric associated the norm p.
It's not obvious, but if (V, p) is a Banach space and q is equivalent to p then (V, q) is also a Banach space.
$\mathbf{R}^n$ is a Banach space, since we've seen that $\mathbf{R}^n$ with the usual norm is complete. If X is a topological space then the set of bounded functions from S to $\mathbf{R}^n$ is a Banach space. Then set of bounded continuous functions is a closed (topological) subspace and closed (topological) subspaces of complete metric spaces are complete, so it is complete. It's also a (linear) subspace, so it's a Banach space. If X is compact then we can drop the word "bounded". For example, the set of continuous functions from an closed finite interval $[a, b]$ to $\mathbf{R}^n$ is a Banach space. The norm is $\|\mathbf{f}\| = \sup_{a \leq t \leq b} \|\mathbf{f}(t)\|$. We can replace the supremum in this case by a maximum.

Bounded linear transformations

If (V, p_V) and (W, p_W) are normed vector spaces and $A \colon V \to W$ is linear then we say that A is *bounded* if there is a $K \geq 0$ such that

$$p_W(A\mathbf{v}) \leq K p_V(\mathbf{v})$$

for all $\mathbf{v} \in V$. K is said to be a *bound* for A.

Warning: The terminology is standard, but it is inconsistent with the way bounded functions from a set to a metric space were defined in Lecture 29, which is also standard. If A is bounded in the sense above and non-zero then A is *not* bounded in the previous sense. The image of a linear transformation is a (linear) subspace and non-zero (linear) subspaces are never bounded.

I've been putting "(linear)" or "(topological)" before the word "subspace" because of another terminological conflict between linear algebra and topology.

For linear transformations I'm also following the standard linear algebra/functional analysis convention of not writing parentheses for function evaluation.

Boundedness and continuity (1/2)

In terms of metrics, if A is bounded and $\mathbf{x}, \mathbf{y} \in V$ then

$$\begin{aligned} d_W(A\mathbf{x}, A\mathbf{y}) &= p_W(A\mathbf{x} - A\mathbf{y}) = p_W(A(\mathbf{x} - \mathbf{y})) \\ &\leq K p_V(\mathbf{x} - \mathbf{y}) = K d_V(\mathbf{x}, \mathbf{y}). \end{aligned}$$

So A is Lipschitz continuous, hence uniformly continuous and hence continuous, and hence continuous at $\mathbf{0}$.
Suppose conversely that A is continuous at $\mathbf{0}$. Choose an $\epsilon > 0$ and then a $\delta > 0$ such that

$$B_V(\mathbf{0}, \delta) \subseteq A^*(B_W(A\mathbf{0}, \epsilon)) = A^*(B_W(\mathbf{0}, \epsilon)).$$

In other words, if $p_V(\mathbf{x}) < \delta$ then $p_W(A\mathbf{x}) < \epsilon$. If $\mathbf{v} \neq \mathbf{0}$ then

$$p_V\left(\frac{\delta}{2p_V(\mathbf{v})}\mathbf{v}\right) = \left|\frac{\delta}{2p_V(\mathbf{v})}\right| p_V(\mathbf{v}) = \frac{\delta}{2} < \delta$$

so

$$p_W\left(A\frac{\delta}{2p_V(\mathbf{v})}\mathbf{v}\right) < \epsilon.$$

Boundedness and continuity (2/2)

$$
\begin{aligned}
p_W(A\mathbf{v}) &= p_W\left(\frac{2p_V(\mathbf{v})}{\delta} A \frac{\delta}{2p_V(\mathbf{v})}\mathbf{v}\right)\\
&= \left|\frac{2p_V(\mathbf{v})}{\delta}\right| p_W\left(\frac{\delta}{2p_V(\mathbf{v})}\mathbf{v}\right)\\
&\leq \frac{2p_V(\mathbf{v})}{\delta}\epsilon\\
&= Kp_V(\mathbf{v})
\end{aligned}
$$

where $K = 2\epsilon/\delta$. This was proved for $\mathbf{v} \neq \mathbf{0}$ but clearly $p_W(A\mathbf{v}) \leq Kp_V(\mathbf{v})$ for $\mathbf{v} = \mathbf{0}$ as well. Therefore A is bounded. So the following five conditions are equivalent for linear functions A:

- A is bounded.
- A is Lipschitz continuous.
- A is uniformly continuous.
- A is continuous.
- A is continuous at $\mathbf{0}$.

The operator norm (1/4)

Suppose $A\colon V \to W$ is bounded and let

$$S = \{K \geq 0 \colon \forall \mathbf{v} \in V \colon p_W(A\mathbf{v}) \leq K p_V(\mathbf{v})\}.$$

S is non-empty because A is bounded. 0 is a lower bound for S so S has an infimum. Let $L = \inf S$. For $\epsilon > 0$ we know that $L + \epsilon$ is not a lower bound for S, so there is a $K \in S$ with $K < L + \epsilon$. $K \in S$ so for all $\mathbf{v} \in V$ we have

$$p_W(A\mathbf{v}) \leq K p_V(\mathbf{v}) \leq (L + \epsilon) p_V(\mathbf{v}).$$

This holds for all $\epsilon > 0$ so

$$p_W(A\mathbf{v}) \leq L p_V(\mathbf{v}).$$

This holds for all $\mathbf{v} \in V$ so $L \in S$. L is therefore not just an infimum but a minimum.

The operator norm (2/4)

The quantity

$$\min\{K \geq 0 \colon \forall \mathbf{v} \in V \colon p_W(A\mathbf{v}) \leq K p_V(\mathbf{v})\},$$

which we just showed exists, is called the *operator norm* of A. In the $\| \quad \|$ notation $\|A\|$ is the least $K \geq 0$ such that

$$\|A\mathbf{v}\| \leq K\|\mathbf{v}\|.$$

The operator norm is a norm.
$\|A\| \geq 0$ and $\|O\| = 0$. If $A \neq O$ then there is a $\mathbf{v} \in V$ such that $A\mathbf{v} \neq \mathbf{0}$. Then $0 < \|A\mathbf{v}\| \leq \|A\|\|\mathbf{v}\|$ so $\|A\| > 0$.

$$\begin{aligned} \|\alpha A\mathbf{v}\| &= \|A(\alpha\mathbf{v})\| \leq \|A\|\|\alpha\mathbf{v}\| \\ &= \|A\||\alpha|\|\mathbf{v}\| = |\alpha|\|A\|\|\mathbf{v}\| \end{aligned}$$

for all $\mathbf{v} \in V$ so $\|\alpha A\| \leq |\alpha|\|A\|$.

The operator norm (3/4)

If $\alpha = 0$ then $\|\alpha A\| = \|O\| = 0 = |\alpha|\|A\|$. Otherwise

$$\begin{aligned}\|A\mathbf{v}\| &= \|\alpha A \alpha^{-1}\mathbf{v}\| \leq \|\alpha A\|\|\alpha^{-1}\mathbf{v}\| \\ &= \|\alpha A\||\alpha^{-1}|\|\mathbf{v}\| = |\alpha|^{-1}\|\alpha A\|\|\mathbf{v}\|.\end{aligned}$$

This holds for all $\mathbf{v} \in V$ so $\|A\| \leq |\alpha|^{-1}\|\alpha A\|$. Combining this with $\|\alpha A\| \leq |\alpha|\|A\|$ gives

$$\|\alpha A\| = |\alpha|\|A\|.$$

Finally,

$$\begin{aligned}\|(A+B)\mathbf{v}\| &= \|A\mathbf{v} + B\mathbf{v}\| \leq \|A\mathbf{v}\| + \|B\mathbf{v}\| \\ &\leq \|A\|\|\mathbf{v}\| + \|B\|\|\mathbf{v}\| = (\|A\| + \|B\|)\|\mathbf{v}\|\end{aligned}$$

for all $\mathbf{v} \in V$ so $\|A + B\| \leq |A\| + \|B\|$.

The operator norm (4/4)

The operator norm is submultiplicative. If A is a bounded linear function from V to W and B is a bounded linear function from U to V then

$$\begin{aligned} \|(AB)\mathbf{v}\| &= \|A(B\mathbf{v})\| \leq \|A\|\|B\mathbf{v}\| \\ &\leq \|A\|\|B\|\|\mathbf{v}\| = (\|A\|\|B\|)\|\mathbf{v}\|. \end{aligned}$$

This holds for all $\mathbf{v} \in U$ so

$$\|AB\| \leq \|A\|\|B\|.$$