MAU22200 Lecture 31

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Completeness of function spaces (1/3)

> *Suppose X is a non-empty topological space and Y is a complete metric space. Then the set of bounded continuous functions from X to Y is a complete metric space.*

Let W be the space of bounded continuous functions from X to Y. We've already seen that W is a metric space. We just need to check completeness.

Suppose $\mathcal{F}$ is a Cauchy filter on W. We want to show that it converges.

As usual there are two steps: find something it ought to converge to, then check that it does.

For each $x \in X$ let $e_x \colon W \to Y$ be the evaluation function $e_x(g) = g(x)$. e_x is uniformly continuous so $e_x^{**}(\mathcal{F})$ is a Cauchy filter on Y. Y is complete so it converges to something in Y. Let $f(x)$ be what $e_x^{**}(\mathcal{F})$ converges to. We want to show that $\mathcal{F}$ converges to f.

Completeness of function spaces (2/3)

$\mathcal{F}$ is Cauchy so for every $\epsilon > 0$ there is an $h \in W$ such that $B_W(h, \epsilon) \in \mathcal{F}$. For all $x \in X$,

$$B_Y(f(x), \epsilon) \in \mathcal{N}_Y(f(x)) \subseteq e_x^{**}(\mathcal{F}).$$

so

$$e_x^*(B_Y(f(x), \epsilon)) \in \mathcal{F}.$$

$$\begin{aligned} g \in e_x^*(B_Y(f(x), \epsilon)) &\Leftrightarrow e_x(g) \in B_Y(f(x), \epsilon) \\ &\Leftrightarrow g(x) \in B_Y(f(x), \epsilon) \\ &\Leftrightarrow d_Y(f(x), g(x)) < \epsilon. \end{aligned}$$

So

$$\{g \in \mathcal{A} \colon d_Y(f(x), g(x)) < \epsilon\} \in \mathcal{F}.$$

Completeness of function spaces (3/3)

$B_W(h,\epsilon) \in \mathcal{F}$ and $\{g \in \mathcal{A} \colon d_Y(f(x), g(x)) < \epsilon\} \in \mathcal{F}$.

$$B_W(h,\epsilon) \cap \{g \in \mathcal{A} \colon d_Y(f(x), g(x)) < \epsilon\} \neq \varnothing$$

so choose a g from the intersection.

$$g \in B_W(h,\epsilon) \cap \{g \in \mathcal{A} \colon d_Y(f(x), g(x)) < \epsilon\}$$

so $d_Y(f(x), g(x)) < \epsilon$ and $d_Y(g(x), h(x)) < \epsilon$. Therefore $d_Y(f(x), h(x)) < 2\epsilon$. This holds for all $x \in X$ so $d_W(f,h) \leq 2\epsilon$. Therefore

$$B_W(h,\epsilon) \subseteq B_W(f, 3\epsilon).$$

$B_W(h,\epsilon) \in \mathcal{F}$ so

$$B_W(f, 3\epsilon) \in \mathcal{F}.$$

This holds for all $\epsilon > 0$ so $\mathcal{F}$ converges to f.
Every Cauchy filter in W is convergent so W is complete.

Equicontinuity

Suppose $\mathcal{T}_X$ is a topology on X and d_Y is a metric on Y. Let $\mathcal{A}$ be a family, i.e. set, of functions from X to Y. We say the family $\mathcal{A}$ is

- *continuous* if for all $f \in \mathcal{A}$ and all $x \in X$ and all $\epsilon > 0$ there is a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(B_Y(f(x), \epsilon))$.
- *uniformly continuous* if for all $f \in \mathcal{A}$ and all $\epsilon > 0$ there is a $U \in \mathcal{O}(x)$ such that for all $x \in X$, $U \subseteq f^*(B_Y(f(x), \epsilon))$.
- *equicontinuous* if for all $x \in X$ and all $\epsilon > 0$ there is a $U \in \mathcal{O}(x)$ such that for all $f \in \mathcal{A}$, $U \subseteq f^*(B_Y(f(x), \epsilon))$.
- *uniformly equicontinuous* if for all $\epsilon > 0$ there is a $U \in \mathcal{O}(x)$ such that for all $f \in \mathcal{A}$ and all $x \in X$, $U \subseteq f^*(B_Y(f(x), \epsilon))$.

Equicontinuity and total boundedness (1/3)

Suppose X is a non-empty compact topological space, Y is a metric space and $\mathcal{A}$ is an equicontinuous set of functions from X to Y. Suppose for each $x \in X$ that

$$I_x = \bigcup_{f \in \mathcal{A}} \{f(x)\}$$

is totally bounded. Then $\mathcal{A}$ is totally bounded.

Suppose $r > 0$ and let $\epsilon = r/4$. Define

$$\mathcal{R} = \{(x, U) \in X \times \mathcal{T}_X : x \in U, U \subseteq f^*(B(f(x), \epsilon))\}.$$

Equicontinuity means that for each $x \in X$ there is a $U \in \mathcal{T}_X$ such that $(x, U) \in \mathcal{R}$. Therefore

$$X = \bigcup_{(x,U) \in \mathcal{R}} U.$$

This is an open cover of the compact set X so has a finite subcover.

Equicontinuity and total boundedness (2/3)

$$X = \bigcup_{j=1}^{m} U_j$$

I_{x_j} is totally bounded for each $1 \le j \le m$ so $\bigcup_{j=1}^{m} I_{x_j}$ is totally bounded. In other words, there are $y_1, \ldots, y_n$ such that

$$\bigcup_{j=1}^{m} I_{x_j} \subseteq \bigcup_{k=1}^{n} B_Y(y_k, \epsilon).$$

For each function $s\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ we consider the set $\mathcal{A}_s$ of $f \in \mathcal{A}$ such that $f(x_j) \in B_Y(y_{s(j)}, \epsilon)$ for all j. The inclusion above shows that each $f \in \mathcal{A}$ belongs to $\mathcal{A}_s$ for at least one s. There may be functions s for which there is no $f \in \mathcal{A}_s$, but let S be the set of s for which there is one. S is finite.

Equicontinuity and total boundedness (3/3)

Any two elements of $\mathcal{A}_s$ are not too far apart. For all $x \in X$ we have $x \in U_j$ for some j and

$$\begin{aligned} d_Y(f(x), g(x)) &\leq d_Y(f(x), f(x_j)) + d_Y(f(x_j), y_j) \\ &\quad + d_Y(y_j, g(x_j)) + d_Y(g(x_j), g(x)) \\ &< \epsilon + \epsilon + \epsilon + \epsilon = r. \end{aligned}$$

$d_Y(f(x), g(x)) < r$ for all $x \in X$ so

$$d_{\mathcal{A}}(f, g) \leq r.$$

Choose an $f_s \in \mathcal{A}_s$ for each $s \in S$. If $g \in \mathcal{A}_s$ then $g \in \bar{B}_{\mathcal{A}}(f_s, r)$. Each $g \in \mathcal{A}$ is in at least one $\mathcal{A}_s$ so

$$\mathcal{A} = \bigcup_{s \in S} \bar{B}_{\mathcal{A}}(f_s, r).$$

There is such a finite cover by closed balls of radius r for each $r > 0$ so $\mathcal{A}$ is totally bounded.

Arzelà-Ascoli

One version of the Arzelà-Ascoli theorem is the following:

> *Suppose X is a compact topological space, Y is a complete metric space, and $\mathcal{A}$ is an equicontinuous set of functions from X to Y. If $\mathcal{A}$ is closed and*
>
> $$I_x = \bigcup_{f \in \mathcal{A}} \{f(x)\}$$
>
> *is totally bounded for each $x \in X$ then $\mathcal{A}$ is compact.*

$\mathcal{A}$ is totally bounded by our second proposition. The set of all bounded continuous functions from X to Y is complete by our first proposition and $\mathcal{A}$ is a closed subset of it so $\mathcal{A}$ is complete by a result from the last lecture. We saw in Lecture 27 that metric spaces are compact if and only if they are complete and totally bounded, so $\mathcal{A}$ is compact.

Another Arzelà-Ascoli

There are a number of versions of the Arzelà-Ascoli Theorem, either more or less general than the one from the previous slide. One useful specialisation is the following:

> *Suppose X is a compact topological space, Y is a compact metric space, and $\mathcal{A}$ is an equicontinuous set of functions from X to Y. If $\mathcal{A}$ is closed then $\mathcal{A}$ is compact.*

Y is complete since it's compact. It's also totally bounded, along with its subsets I_x for all $x \in X$. So all the hypotheses of the previous version of the Arzelà-Ascoli Theorem are satisfied.