MAU22200 Lecture 30

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Cauchy filters and uniform continuity

If X and Y are metric spaces, $f : X \to Y$ is uniformly continuous and $\mathcal{F}$ is a Cauchy filter on X then $f^{**}(\mathcal{F})$ is a Cauchy filter on Y.

Proof: Suppose $\epsilon > 0$. f is uniformly continuous so there is a $\delta > 0$ such that

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon))$$

for all $x \in X$. $\mathcal{F}$ is Cauchy so $B(x, \delta) \in \mathcal{F}$ for some $x \in X$. Therefore

$$f^*(B(f(x), \epsilon)) \in \mathcal{F}$$

In other words,

$$B(f(x), \epsilon) \in f^{**}(\mathcal{F})$$

So $f^{**}(\mathcal{F})$ is Cauchy.

Closed subsets of complete spaces

Suppose X is a complete metric space and A is a closed subset of X. Then A is a complete metric space.

Sketch of proof: $i: A \to X$, defined by $i(x) = x$, is uniformly continuous. If $\mathcal{F}$ is a Cauchy filter on A then $i^{**}(\mathcal{F})$ is a Cauchy filter on X. X is complete so there is a $z \in X$ such that $\mathcal{N}_X(z) \subseteq i^{**}(\mathcal{F})$.
If $z \notin A$ then $X \setminus A \in \mathcal{N}_X(z)$. Then $X \setminus A \in i^{**}(\mathcal{F})$ and $\varnothing = i^*(X \setminus A) \in \mathcal{F}$. This is impossible so $z \in A$.
$\mathcal{F}$ converges to z, i.e for all $V \in \mathcal{N}_A(z)$ we have $V \in \mathcal{F}$. See the notes for the proof.
Every Cauchy filter in A converges, so A is complete.

The Banach Fixed Point Theorem

Suppose that X is a non-empty complete metric space and that there is a $c < 1$ such that $\varphi \colon X \to X$ satisfies

$$d(\varphi(x), \varphi(y)) \leq cd(x, y)$$

for all $x, y \in X$. Then there is a unique $z \in X$ such that $\varphi(z) = z$.

A function φ satisfying the condition above is called a *contraction mapping* and the Banach Fixed Point Theorem is sometimes called the Contraction Mapping Principle. A point z such that $\varphi(z) = z$ is called a *fixed point* of φ.
For any $\epsilon > 0$ let $\delta = \epsilon$ and observe that if $d(x, y) < \delta$ then $d(\varphi(x), \varphi(y)) \leq c\delta < \epsilon$ so φ is uniformly continuous and hence continuous.

Proof (1/3)

If $\varphi(w) = w$ and $\varphi(z) = z$ then

$$d(w, z) = d(\varphi(w), \varphi(z)) \le cd(w, z)$$

so

$$(1 - c)d(w, z) \le 0$$

and $d(w, z) \le 0$. d is a metric so $d(w, z) \ge 0$ so $d(w, z) = 0$ and $w = z$. This shows uniqueness, i.e. that there is at most one fixed point.

To show existence, pick $a \in X$ and define

$$\alpha_0 = a, \qquad \alpha_{j+1} = \varphi(\alpha_j).$$

By induction on n, if $m \le n$ then

$$d(\alpha_m, \alpha_n) \le \frac{c^m - c^n}{1 - c} d(\alpha_0, \alpha_1).$$

Proof (2/3)

$$d(\alpha_m, \alpha_n) \leq \frac{c^m - c^n}{1 - c} d(\alpha_0, \alpha_1)$$

for $m \leq n$ and

$$d(\alpha_m, \alpha_n) \leq \frac{c^n - c^m}{1 - c} d(\alpha_0, \alpha_1)$$

for $n \leq m$ so

$$d(\alpha_m, \alpha_n) \leq \frac{c^{\min(m,n)} - c^{\max(m,n)}}{1 - c} d(\alpha_0, \alpha_1) < \frac{c^{\min(m,n)}}{1 - c} d(\alpha_0, \alpha_1)$$

for all m, n. For any $\epsilon > 0$ there is a k such that $d(\alpha_m, \alpha_n) < \epsilon$ for all $m, n \geq k$. In other words, α is a Cauchy sequence.

Proof (3/3)

α is a Cauchy sequence in X and X is complete so α converges. Let z be its limit. $\alpha_{j+1} = \varphi(\alpha_j)$ so

$$\lim_{j\to\infty} \alpha_{j+1} = \lim_{j\to\infty} \varphi(\alpha_j).$$

φ is continuous so

$$\lim_{j\to\infty} \varphi(\alpha_j) = \varphi\left(\lim_{j\to\infty} \alpha_j\right) = \varphi(z).$$

$$\lim_{j\to\infty} \alpha_{j+1} = \lim_{j\to\infty} \alpha_j = z.$$

So $\varphi(z) = z$.

Spaces of bounded functions

If X is a non-empty set and (Y, d_Y) is a metric space then we say that $f : X \to Y$ is bounded if $f_*(X)$ is a bounded subset of Y. If f and g are bounded then there are $y, z \in Y$ and $r, s > 0$ such that $f_*(X) \subseteq B_Y(y, r)$ and $g_*(X) \subseteq B_Y(z, s)$. If $x \in X$ then

$$d_Y(f(x), g(x)) \leq d_Y(f(x), y)+d_Y(y, z)+d_Y(z, g(x)) < r+d_Y(y, z)+s.$$

The set of $d_Y(f(x), g(x))$ for $x \in X$ is thus bounded from above, so $\sup_{x\in X} d_Y(f(x), g(x))$ exists.

$$d(f, g) = \sup_{x\in X} d_Y(f(x), g(x))$$

is a metric on the space of bounded functions from X to Y. To prove the triangle inequality, for each $x \in X$

$$d_Y(f(x), h(x)) \leq d_Y(f(x), g(x))+d_Y(g(x), h(x)) \leq d(f, g)+d(g, h)$$

so $d(f, h) = \sup_{x\in X} d_Y(f(x), h(x)) \leq d(f, g) + d(g, h)$.

Spaces of bounded continuous functions

If $(X, \mathcal{T})$ is a topological space then the set of bounded continuous functions from X to Y is a closed subset of the space of bounded functions from X to Y.
Proof: Suppose f is bounded but not continuous, i.e that there is an $x \in X$ and an $\epsilon > 0$ such that for all $U \in \mathcal{O}(x)$ we have $U \nsubseteq f^*(B_Y(f(x), \epsilon))$. In other words, there is a $y \in U$ such that $f(y) \notin B_Y(f(x), \epsilon)$, i.e. $d_Y(f(x), f(y)) \geq \epsilon$. If $g \in B(f, \epsilon/3)$ then $d_Y(f(x), g(x)) < \epsilon/3$ and $d_Y(f(y), g(y)) < \epsilon/3$. So $d_Y(g(x), g(y)) \geq \epsilon/3$. In other words, $y \in U$ but $g(y) \notin B_Y(g(x), \epsilon/3)$. So $U \nsubseteq g^*(B_Y(g(x), \epsilon/3))$. Since this holds for all $U \in \mathcal{O}(x)$ we see that g is not continuous. About every discontinuous function there is a ball consisting of discontinuous functions, so the set of discontinuous functions is open. The set of continuous functions is therefore closed.