MAU22200 Lecture 29

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Completion

Suppose (X, d_X) *is a metric space. Let* $\mathbf{X}$ *be the set of minimal Cauchy filters on* X*. Define* $d_{\mathbf{X}} \colon \mathbf{X} \times \mathbf{X} \to \mathbf{R}$ *as follows. If* $\mathcal{F}, \mathcal{G} \in \mathbf{X}$ *then let* $\mathcal{H}$ *be the product of* $\mathcal{F}$ *and* $\mathcal{G}$*. Then* $\mathcal{I} = d_X^{**}(\mathcal{H})$ *is a Cauchy filter on* $\mathbf{R}$*.* $\mathbf{R}$ *is a complete metric space so there is a unique* $z \in \mathbf{R}$ *such that* $\mathcal{I}$ *converges to* z*. We define* $d_{\mathbf{X}}(\mathcal{F}, \mathcal{G}) = z$*. Then* $d_{\mathbf{X}}$ *is a metric on* $\mathbf{X}$*. The function* $i \colon X \to \mathbf{X}$ *defined by* $i(x) = \mathcal{N}(x)$ *satisfies*

$$d_{\mathbf{X}}(i(x), i(y)) = d_X(x, y)$$

for all $x, y \in X$*. Also,* $(\mathbf{X}, d_{\mathbf{X}})$ *is complete.*

We still need to show:

- If $\mathcal{F} \neq \mathcal{G}$ then $d_{\mathbf{X}}(\mathcal{F}, \mathcal{G}) > 0$.
- $d_{\mathbf{X}}(i(x), i(y)) = d_X(x, y)$.
- $(\mathbf{X}, d_{\mathbf{X}})$ is a complete metric space.

$\mathcal{F} \neq \mathcal{G} \Rightarrow d_{\mathbf{X}}(\mathcal{F}, \mathcal{G}) > 0$

By a lemma from last lecture, if $\mathcal{F}$, $\mathcal{G}$ are minimal Cauchy filters and $\mathcal{F} \neq \mathcal{G}$ then there are $x, y \in X$ and $r > 0$ such that $B_X(x, r) \in \mathcal{F}$, $B_X(y, r) \in \mathcal{G}$ and $d_X(x, y) \geq 3r$. If $s \in B(x, r)$ and $t \in B(y, r)$ then

$$3r \leq d_X(x, y) \leq d_X(x, s) + d_X(s, t) + d_X(t, y) < r + d_X(s, t) + r$$

so $d_X(s, t) > r$. From our characterisation of $\mathcal{I}$ from last time we get

$$(r, +\infty) \in \mathcal{I}$$

and

$$d_X(\mathcal{F}, \mathcal{G}) \in [r, +\infty).$$

i.e. $d_X(\mathcal{F}, \mathcal{G}) \geq r$. This holds for some $r > 0$ so

$$d_X(\mathcal{F}, \mathcal{G}) > 0.$$

$d_{\mathbf{X}}(i(x), i(y)) = d_X(x, y)$

$i \colon X \to \mathbf{X}$ was defined by $i(x) = \mathcal{N}(x)$. This makes sense because $\mathcal{N}(x)$ is a minimal Cauchy filter on X.
$B(x, r) \in i(x)$ and $B(y, r) \in i(y)$ so

$$|d_{\mathrm{X}}(i(x), i(y)) - d_X(x, y)| < 2r$$

for all $r > 0$. Therefore

$$d_{\mathrm{X}}(i(x), i(y)) = d_X(x, y).$$

$(\mathbf{X}, d_{\mathbf{X}})$ is complete (1/4)

We need to show that if $\mathfrak{F}$ is a Cauchy filter in $\mathbf{X}$ then it is a convergent filter.
The first step is to find something $\mathfrak{F}$ might plausibly converge to and the second step is to show that $\mathfrak{F}$ does in fact converge to it.
$\mathfrak{F}$ is Cauchy so for each $r > 0$ there's a $\mathcal{G}$ such that

$$B_{\mathrm{X}}(\mathcal{G}, r/4) \in \mathfrak{F}.$$

This $\mathcal{G}$ is also Cauchy so there's an $x \in X$ such that

$$B(x, r/4) \in \mathcal{G}.$$

The factor of $1/4$ is there for later convenience. The $\mathcal{G}$ and x depend on r so I'll write them as $\mathcal{G}(r)$ and $x(r)$.
I claim that $x \colon (0, +\infty) \to X$ is a Cauchy net, if I take the order relation $\geq$ on $(0, +\infty)$.

$(\mathbf{X}, d_{\mathbf{X}})$ is complete (2/4)

Suppose $q \le r$.

$$B_X(\mathcal{G}(q), q/4) \in \mathfrak{F}, \qquad B_X(\mathcal{G}(r), r/4) \in \mathfrak{F}$$

so

$$d_X(\mathcal{G}(q), \mathcal{G}(r)) < q/4 + r/4 \le r/2.$$

$$B_X(x(q), q/4) \in \mathcal{G}(q), \qquad B_X(x(r), r/4) \in \mathcal{G}(r)$$

so

$$\begin{aligned} d_X(x(q), x(r)) &\le d_X(x(q), s) + d_X(s, t) + d_X(t, x(r)) \\ &< q/4 + d_X(s, t) + r/4 \le d_X(s, t) + r/2. \end{aligned}$$

$$d_X(s, t) \in (d_X(x(q), x(r)) - r/2, +\infty).$$

$$d_X(\mathcal{G}(q), \mathcal{G}(r)) \in (d_X(x(q), x(r)) - r/2, +\infty).$$

$(\mathbf{X}, d_{\mathbf{X}})$ is complete (3/4)

$$d_X(\mathcal{G}(q), \mathcal{G}(r)) > d_X(x(q), x(r)) - r/2.$$

$$d_X(x(q), x(r)) < d_X(\mathcal{G}(q), \mathcal{G}(r)) + r/2.$$

$$d_X(\mathcal{G}(q), \mathcal{G}(r)) < r/2.$$

$$d_X(x(q), x(r)) < r.$$

If $q, r \geq r$ then $d_X(x(q), x(r)) < r$. If $q \geq r$ then $d_X(x(q), x(r)) < r$ i.e $x(q) \in B_X(x(r), r)$ for all $q \leq r$. So $B_X(x(r), r)$ is contained in the tail filter of x. For each $r > 0$ there's a ball of radius r in the tail filter, so the tail filter is a Cauchy filter. There is a minimal Cauchy filter which contains the tail filter. Call it $\mathcal{F}$. I claim that $\mathfrak{F}$ converges to $\mathcal{F}$.

$(\mathbf{X}, d_{\mathbf{X}})$ is complete (4/4)

$\mathcal{F}$ is Cauchy so there is a $y \in X$ with $B_X(y, r) \in \mathcal{F}$. $B_X(x(r), r)$ belongs to the tail filter and $\mathcal{F}$ is contained in the tail filter so $B_X(x(r), 3r) \in \mathcal{F}$.

$$B_X(x(r), r/4) \subseteq B_X(x(r), 3r)$$

and $B_X(x(r), r/4) \in \mathcal{G}(r)$ so

$$B_X(x(r), 3r) \in \mathcal{G}(r).$$

This and $B_X(x(r), 3r) \in \mathcal{F}$ imply

$$d_x(\mathcal{F}, \mathcal{G}(r)) < 9r$$

and

$$B(\mathcal{G}(r), r) \subseteq B_X(\mathcal{F}, 10r).$$

$B_X(\mathcal{G}(r), r) \in \mathfrak{F}$ so $B_X(\mathcal{F}, 10r) \in \mathfrak{F}$. This holds for all $r > 0$ so $\mathfrak{F}$ converges $\mathcal{F}$. Thus $(\mathbf{X}, d_X)$ is complete, as promised.

Interpretation

The space $(\mathbf{X}, d_{\mathcal{X}})$ is called the *completion* of (X, d_X).
$(i_*(X), d_{\mathrm{X}})$ is a metric space and i is a bijection from $X \to i_*(X)$ which preserves the metric. One can use it to regard X as a subset of $\mathbf{X}$.
If X is not complete then Cauchy filters may not converge, but there's a larger metric space where they do.
We saw some examples already, e.g. the filters associated to the sequences $1/2^n$ in $(0, +\infty)$ or $\sum_{j=0}^{n} 1/j!$ in $\mathbf{Q}$. The completion of $(0, +\infty)$ is $[0, +\infty)$ while the completion of $\mathbf{Q}$ is $\mathbf{R}$.