MAU22200 Lecture 28

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Product filters

Last time we saw that if $\mathcal{F}$ and $\mathcal{G}$ are filters on X such that $U \cap V \neq \varnothing$ then there is a smallest filter $\mathcal{H}$ on X such that $\mathcal{F} \subseteq \mathcal{H}$ and $\mathcal{G} \subseteq \mathcal{H}$. Explicitly, $\mathcal{H}$ is the set of $W \in \wp(X)$ such that there are $U, V \in \mathcal{F}$ such that $U \cap V \subseteq W$.
Similarly, if $\mathcal{F}$ is a filter on X and $\mathcal{G}$ is a filter on Y then there is a natural filter $\mathcal{H}$ on $X \times Y$. Explicitly, $\mathcal{H}$ is the set of $W \in \wp(X \times Y)$ such that there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that $U \times V \subseteq W$. One way to get this is to apply the previous construction to the filters $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ on $X \times Y$ where $\tilde{U} \in \tilde{\mathcal{F}}$ iff there's a $U \in \mathcal{F}$ such that $U \times Y \subseteq \tilde{U}$ and $\tilde{V} \in \tilde{\mathcal{G}}$ iff there's a $V \in \mathcal{G}$ such that $X \times V \subseteq \tilde{V}$. The condition $\tilde{U} \cap \tilde{V} \neq \varnothing$ is automatically satisfied. In the notes I just construct $\mathcal{H}$ directly. $\mathcal{H}$ is called the *product* of $\mathcal{F}$ and $\mathcal{G}$. As an example, if $\mathbf{w} \in \mathbf{R}^m$ and $\mathbf{z} \in \mathbf{R}^n$ then the product of $\mathcal{N}(\mathbf{w})$ and $\mathcal{N}(\mathbf{z})$ is the filter $\mathcal{N}((\mathbf{w}, \mathbf{z}))$ on $\mathbf{R}^{m+n}$.

Round filters, two more lemmas

A filter $\mathcal{F}$ on a metric space is called *round* if for all $A \in \mathcal{F}$ there is an $r > 0$ such that if $B(x, r) \in \mathcal{F}$ then $B(x, r) \subseteq A$.

Round Cauchy filters and minimal Cauchy filters are the same thing. See the notes for a proof.

If $\mathcal{F}$ is a filter on a metric space X and $B(x, p), B(y, q) \in \mathcal{F}$ then $d(x, y) < p + q$.

Proof: There is a $z \in B(x, p) \cap B(y, q)$ so $d(x, y) \leq d(x, z) + d(z, y) < p + q$.

If $\mathcal{F}$ and $\mathcal{G}$ are minimal Cauchy filters on X such that $\mathcal{F} \neq \mathcal{G}$ then there are $x, y \in X$ and $r > 0$ such that $B(x, r) \in \mathcal{F}$, $B(y, r) \in \mathcal{G}$, and $d(x, y) \geq 3r$.

See the notes for a proof. It uses the fact that minimal Cauchy filters are round.

Completion

The main point of Section 4.6 is the last theorem:

Suppose (X, d_X) is a metric space. Let $\mathbf{X}$ be the set of minimal Cauchy filters on X. Define $d_{\mathbf{X}}\colon \mathbf{X} \times \mathbf{X} \to \mathbf{R}$ as follows. If $\mathcal{F}, \mathcal{G} \in \mathbf{X}$ then let $\mathcal{H}$ be the product of $\mathcal{F}$ and $\mathcal{G}$. Then $\mathcal{I} = d_X^{**}(\mathcal{H})$ is a Cauchy filter on $\mathbf{R}$. $\mathbf{R}$ is a complete metric space so there is a unique $z \in \mathbf{R}$ such that $\mathcal{I}$ converges to z. We define $d_{\mathbf{X}}(\mathcal{F}, \mathcal{G}) = z$. Then $d_{\mathbf{X}}$ is a metric on $\mathbf{X}$. The function $i\colon X \to \mathbf{X}$ defined by $i(x) = \mathcal{N}(x)$ satisfies

$$d_{\mathbf{X}}(i(x), i(y)) = d_X(x, y)$$

for all $x, y \in X$. Also, $(\mathbf{X}, d_{\mathbf{X}})$ is complete.

Unwrapping definitions (1/2)

How are you to understand statements like "$\mathcal{I} = d_X^{**}(\mathcal{H})$ where $\mathcal{H}$ is the product of $\mathcal{F}$ and $\mathcal{G}$."?

We understand sets by identifying their members. For a definition like $T_m = \{n \in \mathbf{N} \colon m \le n\}$ that's easy; $n \in T_m$ if and only if $n \in \mathbf{N}$ and $m \le n$. For a definition like the one above you have to do more work. Do it one step at a time, not all at once!

- $W \in \mathcal{I}$ if and only if $d_X^*(W) \in \mathcal{H}$.
- $d_X^*(W) \in \mathcal{H}$ if and only if there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that that $U \times V \subseteq d_X^*(W)$.
- $U \times V \subseteq d_X^*(W)$ if and only if for all $(s, t) \in U \times V$ implies $(s, t) \in d_X^*(W)$.
- $(s, t) \in d_X^*(W)$ if and only if $d_X(s, t) \in W$ for all $s \in U$ and $t \in V$.

So $W \in \mathcal{I}$ if and only if there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that if $s \in U$ and $t \in V$ then $d_X(s, t) \in W$.

Unwrapping definitions (2/2)

$W \in \mathcal{I}$ if and only if there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that if $s \in U$ and $t \in V$ then $d_X(s,t) \in W$.
Is this clear? Probably not, but at least it's something you can imagine checking. For more insight, consider special cases.
Suppose $\mathcal{F}$ and $\mathcal{G}$ are Cauchy, so we know that there are $x, y \in X$ such that $B_X(x,r) \in \mathcal{F}$ and $B_X(y,r) \in \mathcal{G}$. Let $U = B_X(x,r)$ and $V = B_X(y,r)$. If $s \in U$ and $t \in V$ then

$$d_X(x,s) < r, \qquad d_X(y,s) < r.$$

So

$$\begin{aligned} d_X(s,t) &\le d_X(s,x) + d_X(x,y) + d_X(y,t) < r + d_X(x,y) + r \\ d_X(x,y) &\le d_X(x,s) + d_X(s,t) + d_X(t,y) < r + d_X(s,t) + r \end{aligned}$$

For $s \in U$, $t \in V$, $d_X(s,t) \in (d_X(x,y) - 2r, d_X(x,y) + 2r)$. So $(d_X(x,y) - 2r, d_X(x,y) + 2r) \in \mathcal{I}$.

Consequences (1/3)

If $\mathcal{F}$ and $\mathcal{G}$ are Cauchy then there are $x, y \in X$ such that $B_X(x, r) \in \mathcal{F}$ and $B_X(y, r) \in \mathcal{G}$. Then, as we just saw, $(d_X(x, y) - 2r, d_X(x, y) + 2r) \in \mathcal{I}$. $(d_X(x, y) - 2r, d_X(x, y) + 2r) = B_R(d_X(x, y), 2r)$.

$$B_R(d_X(x, y), 2r) \in \mathcal{I}.$$

So $\mathcal{I}$ is Cauchy, as promised in the statement of the theorem. $\mathcal{I}$ is a filter on $\mathbf{R}$ and $\mathbf{R}$ is complete so $\mathcal{I}$ converges, also as promised. $d_X(\mathcal{F}, \mathcal{G})$ was defined as what it converges to.

$$B_X(d_X(\mathcal{F}, \mathcal{G}), r) \in \mathcal{N}(d_X(\mathcal{F}, \mathcal{G})) \subseteq \mathcal{I}.$$

By one of the lemmas,

$$d_R(d_X(\mathcal{F}, \mathcal{G}), d_X(x, y)) < 3r, \quad \text{i.e.} \quad |d_X(\mathcal{F}, \mathcal{G}) - d_X(x, y)| < 3r.$$

Consequences (2/3)

If $B_X(x,r) \in \mathcal{F}$ and $B_X(y,r) \in \mathcal{G}$ then

$$|d_{\mathbf{X}}(\mathcal{F},\mathcal{G}) - d_X(x,y)| < 3r.$$

Most of what we want to know about $(\mathbf{X}, d_{\mathbf{X}})$ follows from this and the properties of (X, d_X).
For example, if $\mathcal{F} = \mathcal{G}$ we can take $x = y$ so $d_X(x,y) = 0$ and $|d_{\mathbf{X}}(\mathcal{F},\mathcal{G})| < 3r$. This holds for all $r > 0$ so $d_{\mathbf{X}}(\mathcal{F},\mathcal{G}) = 0$.

$$\mathcal{F} = \mathcal{G} \quad \Rightarrow \quad d_{\mathbf{X}}(\mathcal{F},\mathcal{G}) = 0.$$

Also $d_X(x,y) = d_X(y,x)$ so

$$|d_{\mathbf{X}}(\mathcal{F},\mathcal{G}) - d_{\mathbf{X}}(\mathcal{G},\mathcal{F})| < 6r.$$

This holds for all $r > 0$ so

$$d_{\mathbf{X}}(\mathcal{F},\mathcal{G}) = d_{\mathbf{X}}(\mathcal{G},\mathcal{F}).$$

Consequences (3/3)

Suppose $B_X(x, r) \in \mathcal{F}$, $B_X(y, r) \in \mathcal{G}$ and $B_X(z, r) \in \mathcal{J}$.

$$d_X(x, z) \leq d_X(x, y) + d_X(x, z).$$

$$|d_X(\mathcal{F}, \mathcal{G}) - d_X(x, y)| < 3r \qquad |d_X(\mathcal{G}, \mathcal{J}) - d_X(x, y)| < 3r$$
$$|d_X(\mathcal{F}, \mathcal{J}) - d_X(x, y)| < 3r$$

$$d_X(\mathcal{F}, \mathcal{J}) \leq d_X(\mathcal{F}, \mathcal{G}) + d_X(\mathcal{G}, \mathcal{J}) + 9r.$$

This holds for all $r > 0$ so

$$d_X(\mathcal{F}, \mathcal{J}) \leq d_X(\mathcal{F}, \mathcal{G}) + d_X(\mathcal{G}, \mathcal{J}).$$

What's left?

Which parts of the theorem remain to be proved?

- If $\mathcal{F} \neq \mathcal{G}$ then $d_X(\mathcal{F}, \mathcal{G}) > 0$. Then we'll know that d_X is a metric on $\mathbf{X}$.
- $d_X(i(x), i(y)) = d_X(x, y)$.
- $(\mathbf{X}, d_X)$ is a complete metric space.