MAU22200 Lecture 27

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Inclusions of filters

Suppose $\mathcal{F}, \mathcal{G}$ are filters on X and $\mathcal{F} \subseteq \mathcal{G}$.

- If X is a topological space and $\mathcal{F}$ is convergent then so is $\mathcal{G}$. Proof: If $\mathcal{N}(z) \subseteq \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{N}(z) \subseteq \mathcal{G}$.
- If X is a metric space and $\mathcal{F}$ is Cauchy then so is $\mathcal{G}$. Proof: If for all $r > 0$ there is an $x > 0$ such that $B(x, r) \in \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{G}$ then for all $r > 0$ there is an $x > 0$ such that $B(x, r) \in \mathcal{G}$.
- If X is a metric space and $\mathcal{F}$ is Cauchy and $\mathcal{G}$ is convergent then $\mathcal{F}$ is convergent. Proof: For each $r > 0$ there is an $x \in X$ such that $B(x, r) \in \mathcal{F}$. $\mathcal{F} \subseteq \mathcal{G}$ so $B(x, r) \in \mathcal{G}$. For some $z \in X$ we have $\mathcal{N}(z) \subseteq \mathcal{G}$ so $B(z, r) \in \mathcal{G}$. By an earlier lemma $B(z, 3r) \in \mathcal{F}$. This holds for all $r > 0$ so $\mathcal{F}$ converges to z by another earlier lemma.

Compact = complete + totally bounded

If X is totally bounded and $\mathcal{F}$ is a filter then there's a Cauchy filter $\mathcal{G}$ with $\mathcal{F} \subseteq \mathcal{G}$.
Proof: Total boundedness means for each $r > 0$ there are finitely many balls of radius $r > 0$ which cover X. We saw earlier that if $U_1, \ldots, U_m$ is a finite cover of X then we can "add" one of the U_j to any filter. More precisely, if $\mathcal{A}$ is a filter then there's a filter $\mathcal{B}$ which contains $\mathcal{A}$ and U_j. Start with $\mathcal{F}$ and add balls of radius 1/2, 1/4, 1/8, etc. Increasing unions of filters are filters so there's one big filter containing $\mathcal{F}$ and all these balls. It contains balls of all positive radii because if $r > 0$ there's an n with $1/2^n < r$ and filters are upward closed.
If X is also complete then the filter $\mathcal{G}$ is convergent. We saw earlier that X is such that every filter is contained in a convergent filter then X is compact. So totally bounded complete metric spaces are compact. Conversely, compact metric spaces are complete and totally bounded.

Some easy lemmas

- If $\mathcal{F}$ is a filter on a metric space X then there is at most one $z \in X$ such that $\mathcal{F}$ converges to z. Proof: We showed ages ago that metric spaces are Hausdorff and that if X is Hausdorff and $\mathcal{F}$ is a filter on X then there is at most one $z \in X$ such that $\mathcal{N}(z) \subseteq \mathcal{F}$.
- The intersection of any non-empty set of filter is a filter. Proof: Check each of the four properties. They're all easy.
- Suppose X is a topological space $z \in X$, and $A \subseteq X$.
  - If $\mathcal{G}$ is a filter on X, $A \in \mathcal{G}$ and $\mathcal{G}$ converges to z then $z \in \overline{A}$.
  - If $z \in \overline{A}$ then there is a filter $\mathcal{G}$ such that $A \in \mathcal{G}$ and $\mathcal{G}$ converges to z.

  This is an analogue for filters of the fact that if f is a net with values in A and $z = \lim f$ then $z \in \overline{A}$ and, conversely, if $z \in \overline{A}$ then there is a net f with values in A such that $\lim f = z$. Only half of the corresponding statements for sequences is correct in general.

Minimal Cauchy filters (1/2)

A Cauchy filter is called minimal if no proper subset of it is a Cauchy filter. Equivalently, $\mathcal{G}$ is minimal if $\mathcal{F} = \mathcal{G}$ for every Cauchy filter $\mathcal{F}$ such that $\mathcal{F} \subseteq \mathcal{G}$. This is the same sense of the word minimal as for partially ordered sets.
Example: Every neighbourhood filter is a minimal Cauchy filter.
Proof: Suppose $\mathcal{F}$ is a Cauchy filter and $\mathcal{F} \subseteq \mathcal{N}(z)$. $\mathcal{N}(z)$ converges to z. By a lemma from the first slide, $\mathcal{F}$ converges to z. In other words $\mathcal{N}(z) \subseteq \mathcal{F}$. So $\mathcal{F} = \mathcal{N}(z)$. $\mathcal{F} = \mathcal{N}(z)$ for every Cauchy filter $\mathcal{F}$ such that $\mathcal{F} \subseteq \mathcal{N}(z)$, as required.
Not every Cauchy filter is minimal. For example, the sets $(0, b)$ for $b > 0$ are a prefilter on $\mathbf{R}$. Its upward closure is a Cauchy filter, but not a minimal Cauchy filter. It's Cauchy because it contains $B(r, r) = (0, 2r)$ for every $r > 0$. It contains $\mathcal{N}(0)$, because every neighbourhood of 0 contains a set of the form $(0, b)$. But it's not equal to $\mathcal{N}(0)$ because $(0, b) \notin \mathcal{N}(0)$.

Minimal Cauchy filters (2/2)

Every Cauchy filter contains a unique minimal Cauchy filter.
Suppose $\mathcal{H}$ is a Cauchy filter. Let $\mathbf{S}$ be the set of Cauchy filters $\mathcal{G}$ such that $\mathcal{G} \subseteq \mathcal{H}$. The minimal Cauchy filter should be $\mathcal{F} = \bigcap_{\mathcal{G} \in \mathbf{S}} \mathcal{G}$. It's certainly a filter and contained in $\mathcal{H}$. If it's a Cauchy filter then it's certainly minimal. What's not obvious is that it's Cauchy.
$\mathcal{H}$ is Cauchy so there is an $x \in X$ with $B(x, r) \in \mathcal{H}$. If $\mathcal{G} \in \mathbf{S}$ then $\mathcal{G} \subseteq \mathcal{H}$ and $\mathcal{G}$ is Cauchy. So there's a $y \in X$ such that $B(y, r) \in \mathcal{G}$. It follows that $B(x, 3r) \in \mathcal{G}$. Note that y depended on the choice of $\mathcal{G}$ but x didn't. $B(x, 3r) \in \mathcal{G}$ for all $\mathcal{G} \in \mathbf{S}$ so $B(x, 3r) \in \mathcal{F}$. $\mathcal{F}$ contains balls of every positive radius so it's Cauchy.

Joining filters

Suppose $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are filters on X, $\mathcal{F} \subseteq \mathcal{H}$, $\mathcal{G} \subseteq \mathcal{H}$. Then $U \cap V \neq \varnothing$ for all $U \in \mathcal{F}$ and $V \in \mathcal{G}$ because $U, V \in \mathcal{H}$ and $\mathcal{H}$ is a filter. So $U \cap V \neq \varnothing$ for all $U \in \mathcal{F}$ and $V \in \mathcal{G}$ is a necessary condition for the existence of a filter $\mathcal{H}$ such that $\mathcal{F} \subseteq \mathcal{H}$ and $\mathcal{G} \subseteq \mathcal{H}$. It's also a sufficient condition. If $\mathcal{F}$ and $\mathcal{G}$ are filters on X such that $U \cap V \neq \varnothing$ for all $U \in \mathcal{F}$ and $V \in \mathcal{G}$ then there is a filter $\mathcal{H}$ such that $\mathcal{F} \subseteq \mathcal{H}$ and $\mathcal{G} \subseteq \mathcal{H}$. We've already seen the special case $\mathcal{G} = \{A, X\}$. In fact there's a smallest such filter. $W \in \mathcal{H}$ if and only if there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that $U \cap V \subseteq W$. For a proof that this works, see the notes.

If X is a metric space and $\mathcal{F}$ and $\mathcal{G}$ are Cauchy then so is $\mathcal{H}$, by a lemma from the first slide.

If $\mathcal{F}$ and $\mathcal{G}$ are minimal Cauchy filters then $\mathcal{F} = \mathcal{G}$, because there's a *unique* minimal filter contained in $\mathcal{H}$. An equivalent statement is that if $\mathcal{F}$ and $\mathcal{G}$ are distinct minimal Cauchy filters then there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that $U \cap V = \varnothing$.