MAU22200 Lecture 26

John Stalker

Trinity College Dublin

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A useful lemma

Suppose (X, d) is a metric space, $\mathcal{F}$ and $\mathcal{G}$ are filters on X, $x, y \in X$ and $r > 0$. If $B(x, r) \in \mathcal{F}$ and $B(y, r) \in \mathcal{F} \cap \mathcal{G}$ then $B(x, 3r) \in \mathcal{G}$.
$B(x, r) \in \mathcal{F}$ and $B(y, r) \in \mathcal{F}$ so

$$B(x, r) \cap B(y, r) \neq \varnothing.$$

So there's a $z \in B(x, r) \cap B(y, r)$. Suppose $w \in B(y, r)$. Then

$$d(x, w) \leq d(x, z) + d(z, y) + d(y, w) < r + r + r$$

so $w \in B(x, 3r)$. Since this holds for all $w \in B(y, r)$ we have

$$B(y, r) \subseteq B(x, 3r).$$

But $B(y, r) \in \mathcal{G}$ and $\mathcal{G}$ is upward closed so $B(x, 3r) \in \mathcal{G}$.

Cauchy filters and nets

A filter $\mathcal{F}$ on a metric space (X, d) is called a *Cauchy filter* if for all $r > 0$ there is an $x \in X$ such that $B(x, r) \in \mathcal{F}$.
A net is said to be a *Cauchy net* if its tail filter is a Cauchy filter.
A sequence which is a Cauchy net is called a *Cauchy sequence*.
Every convergent filter is a Cauchy filter, since $B(z, r) \in \mathcal{N}(z)$ and hence $B(z, r) \in \mathcal{F}$ if $\mathcal{N}(z) \subseteq \mathcal{F}$. Also, every convergent net is a Cauchy net and every convergent sequence is a Cauchy sequence.
A net $f : D \to X$ is Cauchy if and only if for every $\epsilon > 0$ there is an $a \in D$ such that $d(f(b), f(c)) < \epsilon$ for all $b, c \in D$ such that $a \preccurlyeq b$ and $a \preccurlyeq c$.
Let $\mathcal{F}$ be the tail filter of f.
Suppose f is a Cauchy net, i.e. that $\mathcal{F}$ is a Cauchy filter, and $\epsilon > 0$. There is an $x \in X$ such that $B(x, \epsilon/2) \in \mathcal{F}$.

Cauchy filters and nets, continued

Recall from last time that W in an element of the tail filter of f if and only if there is an $a \in D$ such that $f(b) \in W$ for all $b \in D$ such that $a \preccurlyeq b$. So there's an $a \in D$ such that $f(b) \in B(x, \epsilon/2)$ if $a \preccurlyeq b$. Also, $f(c) \in B(x, \epsilon/2)$ if $a \preccurlyeq c$. If $a \preccurlyeq b$ and $a \preccurlyeq c$ then

$$d(b, c) \leq d(a, b) + d(a, c) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, suppose for every $\epsilon > 0$ there is an $a \in D$ such that $d(f(b), f(c)) < \epsilon$ for all $b, c \in D$ such that $a \preccurlyeq b$ and $a \preccurlyeq c$. Specialise to $c = a$. For every $\epsilon > 0$ there is an $a \in D$ such that $d(f(b), f(a)) < \epsilon$ for all $b \in D$ such that $a \preccurlyeq b$. Then $f(b) \in B(f(a), \epsilon)$ if $a \preccurlyeq b$. So $B(f(a), \epsilon) \in \mathcal{F}$. For all $\epsilon > 0$ there is an x such that $B(x, \epsilon) \in \mathcal{F}$. So $\mathcal{F}$ is a Cauchy filter and f is a Cauchy net.

Comments, examples

The image of a Cauchy sequences is always bounded because all but finitely many elements in the sequence lie in a ball.
The image of a Cauchy net doesn't have to be bounded.
We've seen that convergent filters, nets, sequences are Cauchy.
The converse doesn't have to hold. $\alpha_n = 1/2^n$ is a Cauchy sequence in $(0, +\infty)$ but not a convergent sequence in $(0, +\infty)$.
It is, of course, convergent in $\mathbf{R}$.
There are other examples.

$$\alpha_n = \sum_{j=0}^{n} \frac{1}{j!}$$

is a Cauchy sequence in $\mathbf{Q}$ but is not a convergent sequence in $\mathbf{Q}$.
It is, of course, convergent in $\mathbf{R}$.
The upward closure of the set of intervals $(0, b)$ for $b > 0$ is a Cauchy filter on $(0, +\infty)$ but is not a convergent filter.

Completeness

A metric space where every Cauchy filter is a convergent filter is called *complete*

The preceding examples show that $(0, +\infty)$ and $\mathbf{Q}$ are not complete.

$\mathbf{R}$ is complete though, as is $\mathbf{R}^n$. Every compact metric space is complete. We can prove all of these facts at once, by proving that if every closed ball in (X, d) is compact then (X, d) is complete. Closed balls in $\mathbf{R}^n$ are compact by the Heine-Borel Theorem. Closed balls in a compact metric space are compact because closed subsets of a compact topological space are always compact.

Proof (1/3)

Suppose $\mathcal{F}$ is a Cauchy filter. Let

$$Q = \{(x, r) \in X \times \mathbf{R} \colon r > 0, B(x, r) \in \mathcal{F}\}$$

Choose some $(y, s) \in Q$. If $(x, r) \in Q$ then $\bar{B}(x, r) \in \mathcal{F}$ because $B(x, r) \subseteq \bar{B}(x, r)$ and $\mathcal{F}$ is upward closed. If $(x_1, r_1), \ldots,$ $(x_m, r_m) \in Q$ then

$$\bigcap_{j=1}^{m} \bar{B}(x_j, r_j) \neq \varnothing.$$

Let $\mathcal{C}$ be the set of sets of the form $\bar{B}(x, r) \cap \bar{B}(y, s)$ for $(x, r) \in Q$. The intersection of finitely many elements of $\mathcal{C}$ is always non-empty. The elements of $\mathcal{C}$ are closed subsets of the compact set $\bar{B}(y, s)$. So the intersection of all elements of $\mathcal{C}$ is non-empty, i.e.

$$\bigcap_{(x,r)\in Q} \left(\bar{B}(x, r) \cap \bar{B}(y, s)\right) \neq \varnothing.$$

Proof (2/3)

$$\bigcap_{(x,r)\in Q} \left(\bar{B}(x,r) \cap \bar{B}(y,s)\right) = \left(\bigcap_{(x,r)\in Q} \bar{B}(x,r)\right) \cap \bar{B}(y,s)$$
$$= \bigcap_{(x,r)\in Q} \bar{B}(x,r)$$

since $(y,s) \in Q$ so

$$\bigcap_{(x,r)\in Q} \bar{B}(x,r) \subseteq \bar{B}(y,s).$$

So there is a $z \in \bigcap_{(x,r)\in Q} \bar{B}(x,r)$. For any given $r > 0$ choose an x such that $B(x,r) \in \mathcal{F}$. There must be one because $\mathcal{F}$ is a Cauchy filter. Then $(x,r) \in Q$ so $z \in \bar{B}(x,r)$. Suppose $y \in B(x,r)$. Then $d(x,y) < r$ and $d(x,z) \leq r$ so $d(y,z) < 2r$. In other words, $y \in B(z,2r)$. This holds for all $y \in B(x,r)$ so

$$B(x,r) \subseteq B(z,2r).$$

Proof (3/3)

$$B(x, r) \subseteq B(z, 2r),$$

$B(x, r) \in \mathcal{F}$, and $\mathcal{F}$ is upward closed, so $B(z, 2r) \in \mathcal{F}$. So for every $r > 0$ we have $B(z, 2r) \in \mathcal{F}$. Therefore $\mathcal{F}$ converges to z by the lemma from the end of the last lecture.
We've taken an arbitrary Cauchy filter on X and shown that it converges, so X is complete.