MAU22200 Lecture 25

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Quick review of filters

A *filter* on X is an $\mathcal{F} \in \wp(\wp(X))$ such that:

- $\mathcal{F} \neq \varnothing$.
- $\varnothing \notin \mathcal{F}$.
- If $A, B \in \mathcal{F}$ then there is a $C \in \mathcal{F}$ such that $C \subseteq A \cap B$.
- If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.

Some consequences of the definition are that $X \in \mathcal{F}$ and if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$. More generally, if $A_1, \ldots, A_m \in \mathcal{F}$ then $\bigcap_{j=1}^{m} A_j \in \mathcal{F}$. It follows that $\bigcap_{j=1}^{m} A_j \neq \varnothing$.
A *prefilter* is a set of sets satisfying the first three conditions above. We can get a filter from a prefilter by taking its upward closure. The upward closure of a prefilter is the smallest filter containing it.

The eventuality filter

A directed set is a pair $(D, \preccurlyeq)$ where D is a (non-empty) set and $\preccurlyeq$ is a relation on D such that

- $a \preccurlyeq a$.
- If $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$.
- For any a and b there is a c such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

It's useful to consider $\tau \colon D \to \wp(D)$ defined by

$$\tau(a) = \{b \in D \colon a \preccurlyeq b\}.$$

$\tau_*(D)$ is a prefilter. Its upward closure is called the *eventuality filter* of $(D, \preccurlyeq)$.
For example, $(\mathbf{N}, \leq)$ is directed set. $\tau(m)$ is the set of integers greater than or equal to m. $\tau_*(\mathbf{N})$ is the set of non-empty sets of non-negative integers which, if they contain an integer also contain all larger integers. The eventuality filter is the set of supersets of such sets, i.e. the sets with finite complements.

Tail filters of nets

A *net* is a function whose domain is a directed set. It's called a *sequence* if that directed set is $(\mathbf{N}, \leq)$. If $f : D \to X$ then the *tail filter* of f is $f^{**}(\mathcal{F})$ where $\mathcal{F}$ is the eventuality filter of $(D, \preccurlyeq)$. This is a filter on X.
More generally, if $f : S \to X$ is a function and $\mathcal{F}$ is a filter on S then $f^{**}(\mathcal{F})$ is a filter on X.
If $(X, \mathcal{T})$ is a topological space and $z \in X$ then a filter $\mathcal{F}$ is said to *converge* to z if

$$\mathcal{N}(z) \subseteq \mathcal{F}.$$

The tail filter has a more explicit description. W is an element of the tail filter of f if and only if there is an $a \in D$ such that $f(b) \in W$ for all $b \in D$ such that $a \preccurlyeq b$. So f converges to z if and only if for all neighbourhoods W of z there is an $a \in D$ such that $f(b) \in W$ for all $b \in D$ such that $a \preccurlyeq b$. This is (nearly) how we defined limits of nets in Chapter 1.

Sequences and subsequences

Suppose $\alpha \colon \mathbf{N} \to X$ is a sequence. The eventuality filter $\mathcal{F}$ for $(\mathbf{N}, \leq)$ is the set of subsets of $\mathbf{N}$ with finite complement. The tail filter $\alpha^{**}(\mathbf{N})$ is the set of subsets $A \subseteq X$ such that $\alpha^*(A) \in \mathcal{F}$. In other words $\alpha_n \in A$ for all but finitely many n.
Consider a subsequence. Its tail filter is the set of subsets $A \subseteq X$ such that $\alpha_{n_k} \in A$ for all but finitely many k. This is a weaker condition, so the tail filter is larger. So subsequences correspond to superfilters! Almost no one calls them superfilters though. Some people use the term "subordinate filter" though.

Random facts about filters

- If $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ is an increasing sequence of filters then $\mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots$ is also filter.
- If $U_1, \ldots, U_m$ cover X, i.e.

$$X = \bigcup_{j=1}^{m} U_m$$

  and $\mathcal{F}$ is a filter on X then there is a $j \in \{1, \ldots, m\}$ such that $U_j \cap V \neq \varnothing$ for all $V \in \mathcal{F}$. Equivalently, there's some j such that the set of sets $U_j \cap V$ for $V \in \mathcal{F}$ is a filter on U_j.
- If U is a set such that $U \cap V \neq \varnothing$ for all $V \in \mathcal{F}$ then the upward closure of the set of sets $U \cap V$ is a filter on X.
- If $U_1, \ldots, U_m$ cover X and $\mathcal{F}$ is a filter on X then there is a filter $\mathcal{G}$ on X and a j such that $U_j \in \mathcal{G}$ and $\mathcal{F} \subseteq G$.

This generalises the fact if $U_1, \ldots, U_m$ cover X then every sequence has a subsequence contained in U_j for some j.

A compactness criterion

Suppose for every filter $\mathcal{F}$ on a topological space X there is a convergent filter $\mathcal{G}$ such that $\mathcal{F} \subseteq \mathcal{G}$. Then X is compact.
This is a substitute for the (false) statement that if every sequence in X has a convergent subsequence then X is compact.
We prove this using the finite intersection criterion.
Suppose $\mathcal{C}$ is a non-empty collection of closed subsets such that every finite intersection is non-empty. Let $\mathcal{E}$ be the set of all such finite intersections. $\mathcal{E} \neq \varnothing$. $\varnothing \notin \mathcal{E}$. If $A, B \in \mathcal{E}$ then $A \cap B \in \mathcal{E}$. So $\mathcal{E}$ is a prefilter. Its upward closure, $\mathcal{F}$, is a filter. By hypothesis, $\mathcal{F} \subseteq \mathcal{G}$ for some convergent $\mathcal{G}$. There is a z such that $\mathcal{N}(z) \subseteq \mathcal{G}$. If $W \in \mathcal{C}$ and $U \in \mathcal{N}(z)$ then $U, W \in \mathcal{G}$. So $U \cap W \neq \varnothing$. $U \cap W \neq \varnothing$ for all $U \in \mathcal{N}(z)$ so $z \in \overline{W}$. But $\overline{W} = W$, so $z \in W$. This holds for all $W \in \mathcal{C}$, so $z \in \bigcap_{W \in \mathcal{C}} W$. Therefore $\bigcap_{W \in \mathcal{C}} W \neq \varnothing$. So X is compact.

Convergence in metric spaces

If (X, d) is a metric space, $z \in X$ and $\mathcal{F}$ is a filter on X then $\mathcal{F}$ converges to z if and only if there is an $n > 0$ such that

$$B(z, nr) \in \mathcal{F}$$

for all $r > 0$.
The "only if" part is easy because $B(z, nr) \in \mathcal{N}(z)$, so we just need to prove the "if".
Suppose then there is such an n. For any $V \in \mathcal{N}(z)$ there is a $\delta > 0$ such that $B(z, \delta) \subseteq V$. Let $r = \delta/n$, so $B(z, nr) \subseteq V$. $B(z, nr) \in \mathcal{F}$ and $\mathcal{F}$ is upward closed so $V \in \mathcal{F}$. This holds for all $V \in \mathcal{N}(z)$ so

$$\mathcal{N}(z) \subseteq \mathcal{F}$$

Therefore $\mathcal{F}$ converges to z.