MAU22200 Lecture 24

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Images of bounded sets under Lipschitz functions

If X and Y are metric spaces, $f: X \to Y$ is Lipschitz continuous and X is bounded then $f_*(X)$ is bounded.
This wouldn't be true if we replaced "Lipschitz continuous" with "continuous" or even "uniformly continuous".
Let (Y, d_Y) be any unbounded metric space. Let $X = Y$, but let d_X be the discrete metric on X. (X, d_X) a bounded metric space.
Let $i: X \to Y$ be the identity function $i(x) = x$, but considered as a function from $(X, d_X) \to (Y, d_Y)$. i is continuous because any function from a space with the discrete metric to any topological space is continuous. i is also uniformly continuous. For any $\epsilon > 0$ let $\delta = 1/2$. $B_X(x, \delta) = \{x\}$ and $B_Y(i(x), \epsilon) = B_Y(x, \epsilon)$.
$i^*(B_Y(i(x), \epsilon) = B_Y(x, \epsilon)$. $x \in B_Y(x, \epsilon)$. So
$B_X(x, \delta) \subseteq i^*(B_Y(i(x), \epsilon)$. Therefore i is uniformly continuous.
There is a different counter-example in the notes.

Proof

If X and Y are metric spaces, $f : X \to Y$ is Lipschitz continuous and X is bounded then $f_*(X)$ is bounded.
Proof: X is bounded so there's an $r > 0$ such that

$$d_X(s, t) \leq r.$$

f is Lipschitz, so there's a $K \geq 0$ with

$$d_Y(f(s), f(t)) \leq K d_X(s, t).$$

$$d_Y(f(s), f(t)) \leq Kr < Kr + 1.$$

$Kr + 1 > 0$ and every pair w, z of points in $f_*(X)$ is of the form $w = f(s)$, $z = f(t)$ for some $s, t \in X$. So $f_*(X)$ is bounded.

Images of totally bounded sets

If X and Y are metric spaces, $f : X \to Y$ is uniformly continuous and X is totally bounded then $f_*(X)$ is totally bounded.
This is no longer true if we replace "uniformly continuous" with "continuous". Let $X = (0, 1)$, $Y = (1, +\infty)$, both with the usual metric, and $f(x) = 1/x$. f is continuous and $f_*(X) = Y$. X is totally bounded. For any $r > 0$ choose $n > \frac{1}{2r}$, and then

$$X = (0, 1) \subseteq \bigcup_{j=1}^{n} B\left(\frac{2j-1}{2n}, r\right).$$

Y is not totally bounded. If $F \in (1, +\infty)$ and $r > 0$ then

$$r + \max_{x \in F} x \in (1, +\infty) \setminus \bigcup_{x \in F} B(x, r)$$

so we can't have $Y \subseteq \bigcup_{x \in F} B(x, r)$.

Proof

If X and Y are metric spaces, $f : X \to Y$ is uniformly continuous and X is totally bounded then $f_*(X)$ is totally bounded.
Proof: Suppose $r > 0$. f is uniformly continuous so there is a $\delta > 0$ such that for all $x \in X$ $B_X(x, \delta) \subseteq f^*(B_Y(f(x), r))$.
(X, d_X) is totally bounded so there are $x_1, \ldots, x_m \in X$ such that $X \subseteq \bigcup_{j=1}^m B_X(x_j, \delta)$.

$$
\begin{aligned}
f_*(X) &\subseteq f_*\left(\bigcup_{j=1}^m B_X(x_j, \delta)\right) \subseteq \bigcup_{j=1}^m f_*\left(B_X(x_j, \delta)\right) \\
&\subseteq \bigcup_{j=1}^m f_*\left(f^*\left(B_Y(f(x_j), r)\right)\right) \subseteq \bigcup_{j=1}^m B_Y(f(x_j), r),
\end{aligned}
$$

because $f_*(f^*(S)) \subseteq S$ for any S. So $f_*(X)$ is totally bounded.

Distances to sets

The metric d gives the distance from a point to a point. We can also define the distance from a point to a set.

$$r(x) = \inf_{y \in A} d(x, y).$$

is the distance from x to A, if $A \neq \emptyset$. It's non-negative, and is zero if and only if $x \in \overline{A}$. It's also Lipschitz continuous, with $K = 1$. It's therefore uniformly continuous and continuous.
Proof: If $x \in \overline{A}$ then $A \cap B(x, \delta) \neq \emptyset$ for all $\delta > 0$. In other words, there is a $y \in A$ such that $d(x, y) < \delta$. So $r(x) < \delta$. This holds for all $\delta > 0$ and $r(x) \geq 0$ so $r(x) = 0$.
Conversely, if $r(x) = 0$ then $r(x) < \delta$ for all $\delta > 0$. So there is a $y \in A$ with $d(x, y) < \delta$, and $A \cap B(x, \delta) \neq \emptyset$. This holds for all $\delta > 0$, so $x \in \overline{A}$.

Proof of Lipschitz continuity

Trick: To show that $|r(s) - r(t)| \leq d(s, t)$ we first show that $|r(s) - r(t)| \leq d(s, t) + \delta$ for all $\delta > 0$.

Note that this trick works only because the inequality we want to prove is non-strict, i.e. $\leq$, rather than strict, i.e. $<$.

There is a $y \in A$ with $d(t, y) < r(t) + \delta$. Otherwise $r(t)$ wouldn't be the infimum of $d(t, y)$ over $y \in A$.

$$d(s, y) \leq d(s, t) + d(t, y) < d(s, t) + r(t) + \delta.$$

So $r(s) \leq d(s, t) + r(t) + \delta$. In other words, $r(s) - r(t) \leq d(s, t) + \delta$. Similarly, $r(t) - r(s) \leq d(s, t) + \delta$. So

$$|r(s) - r(t)| \leq d(s, t) + \delta.$$

This holds for all δ, so

$$|r(s) - r(t)| \leq d(s, t).$$

This is the Lipschitz condition with $K = 1$.

A Urysohn-like result

If A and B are non-empty closed subsets in a metric space X and $A \cap B \neq \varnothing$ then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$.

$$f(x) = \frac{r_A(x)}{r_A(x) + r_B(x)}$$

works. The denominator has no zeroes because A and B are disjoint and closed.

Metric spaces are normal

Suppose A and B are non-empty closed subsets in a metric space X and $A \cap B \neq \varnothing$. Let f be as on the previous slide and

$$V = f^*([0, 1/3)), \quad W = f^*((2/3, 1]).$$

Then $A \subseteq V$ and $B \subseteq W$. Also V and W are open, because f is continuous. And $V \cap W = \varnothing$.

If A and B are non-empty closed subsets in a metric space X and $A \cap B \neq \varnothing$ then there are open subsets V and W such that $A \subseteq V$, $B \subseteq W$ and $V \cap W = \varnothing$. This is almost the definition of a normal space. We just need to drop the hypothesis that A and B are non-empty. But we can, since the conclusion follows trivially if either is empty.