MAU22200 Lecture 23

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8 November 2021

Boundedness (definition) (1/2)

Not every notion about metric spaces is a specialisation of one about topological spaces. Suppose (X, d) is a metric space and X is non-empty. The following conditions are equivalent.

- There is an $r > 0$ such that $d(x, y) \leq r$ for all $x, y \in X$.
- There is an $r > 0$ such that $d(x, y) < r$ for all $x, y \in X$.
- There is an $r > 0$ such that $X = B(x, r)$ for all $x \in X$.
- There is an $r > 0$ such that $X = \bar{B}(x, r)$ for all $x \in X$.
- For all $x \in X$ there is an $r > 0$ such that $X = \bar{B}(x, r)$.
- For all $x \in X$ there is an $r > 0$ such that $X = B(x, r)$.
- There is an $x \in X$ and an $r > 0$ such that $X = B(x, r)$.
- There is an $x \in X$ and an $r > 0$ such that $X = \bar{B}(x, r)$.

If A satisfies any (and hence all) of these then it is called *bounded*. A subset is called bounded if it's bounded as a metric space with the restriction of d as its metric.

Boundedness (definition) (2/2)

Suppose (X, d) is a metric space and $A \in \wp(X)$ is non-empty. The following conditions are equivalent.

- A is bounded, i.e. t here is an $r > 0$ such that $d(x, y) \leq r$ for all $x, y \in A$.
- There is an $r > 0$ such that $d(x, y) < r$ for all $x, y \in A$.
- There is an $r > 0$ such that $A \subseteq B(x, r)$ for all $x \in A$.
- There is an $r > 0$ such that $A \subseteq \bar{B}(x, r)$ for all $x \in A$.
- For all $x \in A$ there is an $r > 0$ such that $A \subseteq \bar{B}(x, r)$.
- For all $x \in A$ there is an $r > 0$ such that $A \subseteq B(x, r)$.
- There is an $x \in A$ and an $r > 0$ such that $A \subseteq B(x, r)$.
- There is an $x \in A$ and an $r > 0$ such that $A \subseteq \bar{B}(x, r)$.

These are more or less the same as before, but with "$\in A$" in place of "$\in X$" and "$A \subseteq$" in place of "$X =$".

Properties, examples

Subsets and finite unions of bounded sets are bounded. Finite sets are bounded.

Infinite unions of bounded sets needn't be bounded. For example $\mathbf{R} = \bigcup_{n \in Z} [n - 1/2, n + 1/2]$. Each $[n - 1/2, n + 1/2]$ is bounded, but $\mathbf{R}$ is not.

For intervals in $\mathbf{R}$, $\varnothing$, $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) are all bounded. $[a, +\infty)$, $(a, +\infty)$, $(-\infty, b)$, $(-\infty, b]$ and $(-\infty, +\infty) = \mathbf{R}$ are all unbounded.

Any set with the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is bounded.

Total boundedness

A metric space (X, d) is called *totally bounded* if for all $r > 0$ there's a finite $F \in \wp(X)$ such that $X = \bigcup_{x \in F} B(x, r)$.

If X is totally bounded then it is bounded, but the converse is false. A set with the discrete metric is always bounded, but is totally bounded only if it is finite.

Compact metric spaces are always totally bounded, since the balls of given radius cover X.

There's also a notion of total boundedness for subsets, with the obvious definition and properties.

Lipschitz and uniform continuity

In addition to continuity we have two stronger notions which apply to functions between metric spaces.
$f : X \to Y$ is called *Lipschitz continuous* if there is a $K \geq 0$ such that $d_Y(f(s), f(t)) \leq K d_X(s, t)$ for all $s, t \in X$.
$f : X \to Y$ is called *uniformly continuous* if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$ we have $B(x, \delta) \subseteq f^*(B(f(x), \epsilon))$.
Note that δ depends only on ϵ, not on x.
Lipschitz continuity implies uniform continuity and uniform continuity implies continuity.
$f : [0, 1] \to [0, 1]$ defined by $f(x) = \sqrt{x}$ is uniformly continuous but not Lipschitz continuous.
$f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$ is continuous but not uniformly continuous.
On the other hand, if X is compact and $f : X \to Y$ is continuous then it must be uniformly continuous.

Continuity on compact sets implies uniform continuity (1/3)

If X is compact and $f : X \to Y$ is continuous then it must be uniformly continuous.

Suppose $\epsilon > 0$. For every $x \in X$ there is a $\delta > 0$ such that

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon/2)).$$

$$X = \bigcup_{\substack{x \in X, \delta > 0 \\ B(x,\delta) \subseteq f^*(B(f(x),\epsilon/2))}} B(x, \delta/2).$$

I.e. the balls $B(x, \delta/2)$ as above form an open cover of X. X is compact, so there are $x_1, \ldots, x_m \in X$ and $\delta_1, \ldots, \delta_m > 0$ such that

$$B(x_j, \delta_j) \subseteq f^*(B(f(x_j), \epsilon/2)) \qquad , X = \bigcup_{j=1}^{m} B(x_j, \delta_j/2).$$

Continuity on compact sets implies uniform continuity (2/3)

Let $\delta = \frac{1}{2}\min_{1\leq j\leq m}\delta_j$. Suppose $x \in X$ and $y \in B(x,\delta)$, i.e.

$$d(x,y) < \delta.$$

$x \in B(x_j, \delta_j/2)$ for some j, i.e.

$$d(x,x_j) < \delta_j/2.$$

$$d(x,y) < \delta_j/2.$$

$$d(x_j,y) < \delta_j.$$

$$d(f(x_j),f(y)) < \epsilon/2, \qquad d(f(x_j),f(x)) < \epsilon/2.$$

Continuity on compact sets implies uniform continuity (3/3)

$$d(f(x_j), f(y)) < \epsilon/2, \qquad d(f(x_j), f(x)) < \epsilon/2.$$

$$d(f(x), f(y)) < \epsilon,$$

so $f(y) \in B(f(x), \epsilon)$ or, equivalently,

$$y \in f^*(B(f(x), \epsilon)).$$

For all $x \in X$ we've shown that if $y \in B(x, \delta)$ then $y \in f^*(B(f(x), \epsilon))$. In other words,

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon)).$$

So f is uniformly continuous.

Properties of Lipschitz and uniformly continuous functions

- Restrictions of Lipschitz continuous functions are Lipschitz continuous.
- Restrictions of uniformly continuous functions are uniformly continuous.
- Compositions of Lipschitz continuous functions are Lipschitz continuous.
- Compositions of uniformly continuous functions are uniformly continuous.

Only the last of these is not obvious.