MAU22200 Lecture 22

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Partitions of unity

Suppose $(X, \mathcal{T})$ is a normal topological space and let $\mathcal{C}$ be the set of closed subsets of X. that $K \colon L \to \mathcal{C}$ and $U \colon L \to \mathcal{T}$ are indexed collections of sets with the following properties.

- $X = \bigcup_{\lambda \in L} K_\lambda$.
- $K(\lambda) \subseteq U(\lambda)$ for all $\lambda \in L$.
- For all $x \in X$ there is $V \in \mathcal{O}(x)$ such that the set $\{\lambda \in L \colon V \cap U(\lambda) \neq \varnothing\}$ is finite.

Then there is a function $g \colon L \times X \to [0, 1]$, continuous in its second argument, satisfying the following conditions.

- For each $\lambda \in L$ the set $\{x \colon g(\lambda, x) > 0\}$ is a subset of $U(\lambda)$.
- For each $x \in X$ we have $\sum_{\lambda \in L} g(\lambda, x) = 1$.

The main use of this proposition is to be able to write

$$\int_X f = \sum_{\lambda \in L} \int_{K(\lambda)} f$$

Metric spaces

Every metric space is a topological space. All the usual definitions and theorems apply. For example, the composition of continuous functions between metric spaces is continuous. But there are some new theorems, and new "definitions", i.e. alternate characterisations of things previously defined.

Mostly these are the same as the old ones with

- open neighbourhoods replaced by balls, and
- nets replaced by sequences.

They're usually proved using the fact that, in a metric space,

- every open ball is an open neighbourhood,
- every open neighbourhood contains an open ball, and
- there is countable collection of balls about a point such that every neighbourhood of it contains one of them, e.g. $B(x, 1/2^k)$.

Closure (1/2)

We know that $x \in \overline{A}$ if and only if $A \cap V \neq \varnothing$ for all $V \in \mathcal{O}(x)$.
We can guess that $x \in \overline{A}$ if and only if $A \cap B(x, r) \neq \varnothing$ for every $r > 0$.
We can prove this as follows:
$x \in \overline{A}$ if $A \cap B(x, r) \neq \varnothing$ for every $r > 0$: If $V \in \mathcal{O}(x)$ then there is an $r > 0$ such that $B(x, r) \subseteq V$. Then $A \cap B(x, r) \subseteq A \cap V$. $A \cap B(x, r) \neq \varnothing$. So $A \cap V \neq \varnothing$. Therefore $x \in \overline{A}$.
$x \in \overline{A}$ only if $A \cap B(x, r) \neq \varnothing$ for every $r > 0$: If $x \in \overline{A}$ then $A \cap V \neq \varnothing$ for all $V \in \mathcal{O}(x)$. $B(x, r) \in \mathcal{O}(x)$ for all $r > 0$. So $A \cap B(x, r) \neq \varnothing$ for all $r > 0$.
A very similar argument shows that $x \in A^\circ$ if and only if there is an $r > 0$ such that $B(x, r) \subseteq A$, or that $x \in \partial A$ if and only if $A \cap B(x, r) \neq \varnothing$ and $(X \setminus A) \cap B(x, r) \neq \varnothing$ for all $r > 0$.

Closure (2/2)

We also know that $x \in \overline{A}$ if and only if there is a net $f : D \to X$ such that $f(a) \in A$ for all $a \in D$ and $\lim f = x$.
We can guess that $x \in \overline{A}$ if and only if there is a sequence $\alpha : \mathbf{N} \to X$ such that $\alpha_n \in A$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} \alpha_n = x$.
Two comments:

- We only expect this if A is a subset of a metric space. I'll have more to say about this in a moment.
- Half of the new proposition follows from the old one, since every sequence is a net, so we need only show the "only if".

If $x \in \overline{A}$ then $A \cap B(x, r) \neq \varnothing$ for all for all $r > 0$, in particular for $r = 1/2^k$. Choose $\alpha_k \in A \cap B(x, 1/2^k)$. Then $\alpha_k \in A$ for all k.
Also, for all $V \in \mathcal{O}(x)$ there is an $m \in \mathbf{N}$ such that $B(x, 1/2^m) \subseteq V$, and $B(x, 1/2^n) \subseteq V$ for all $n \geq m$. This is (nearly) the definition of the limit of a sequence in a topological space, so $\lim_{n\to\infty} \alpha_n = x$.

Subspace topologies

There's a slight ambiguity above. Suppose (X, d_X) is a metric space and $A \in \wp(X)$. What is the appropriate topology to put on A? We have two candidates:

- X is a metric space and has a topology $\mathcal{T}_X$ of open sets. A is a subset, and so has a subspace topology. Call this topology $\mathcal{T}_1$.
- The restriction of d_X to A, d_A, is a metric on A. So (A, d_A) is a metric space. The set of open sets of A with respect to d_A is a topology. Call this topology $\mathcal{T}_2$.

It $\mathcal{T}_1 = \mathcal{T}_2$? Yes, but this isn't obvious, and requires a proof. See the notes for details.

Continuous functions (1/2)

When we have more than one space there are multiple ways to specialise.
Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. $f : X \to Y$ is continuous at $x \in X$ if and only if $f^*(W) \in \mathcal{N}(x)$ for all $W \in \mathcal{N}(f(x))$.
The first step is to rephrase this in terms of open neighbhourhoods. $f : X \to Y$ is continuous at $x \in X$ if and only if for every $V \in \mathcal{O}(f(x))$ there is a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(V)$.
We can guess that if (X, d_X) and (Y, d_Y) are metric spaces then f is continuous at x if and only if for all $\epsilon > 0$ there is a $\delta > 0$ such that $B_X(x, \delta) \subseteq f^*(B_Y(f(x), \epsilon))$.
This is correct, but it's not the only useful specialisation. If (X, d_X) is a metric spaces and $(Y, \mathcal{T}_Y)$ is a topological space then f is continuous at x if and only if for every $V \in \mathcal{O}(f(x))$ there is a $\delta > 0$ such that $B_X(x, \delta) \subseteq f^*(V)$. There's a similar statement when $(X, \mathcal{T}_X)$ is a topological space and (Y, d_Y) is a metric space.

Continuous functions (2/2)

Proof of the version with two metric spaces:
Suppose f is continuous at x and $\epsilon > 0$. $B_Y(f(x), \epsilon)$ in $\mathcal{O}(f(x))$. So there is a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(B_Y(f(x), \epsilon))$. There is then a $\delta > 0$ such that $B_X(x, \delta) \subseteq U$. Then $B_X(x, \delta) \subseteq f^*(B_Y(f(x), \epsilon))$.
Suppose that for all $\epsilon > 0$ there is a $\delta > 0$ such that $B_X(x, \delta) \subseteq f^*(B_Y(f(x), \epsilon))$. Suppose $V \in \mathcal{O}(f(x))$. Then there is an $\epsilon > 0$ such that $B_Y(f(x), \epsilon) \subseteq V$. Then $f^*(B_Y(f(x), \epsilon)) \subseteq f^*(V)$. There is a $\delta > 0$ such that $B_X(x, \delta) \subseteq f^*(B_Y(f(x), \epsilon))$. Then $B_X(x, \delta) \subseteq f^*(V)$. $B_X(x, \delta) \in \mathcal{O}(x)$. So f is continuous at x.